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Identifying Restrained Domination in the Corona of the Two Connected Graphs

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Abstract: Let G be a connected simple graph. A subset S of V(G) is a dominating set of G if for every $v \in V(G) \setminus S$, there exists $x \in S$ such that $xv \in E(G)$. An identifying code S of a graph G is a dominating set $S \subseteq V(G)$ such that for every $v \in V(G)$, $N_G[v] \cap S$ is distinct. An identifying code of a graph G is an identifying restrained dominating set if every vertex not in S is adjacent to a vertex in S and to a vertex in $V(G) \setminus S$. Alternately, an identifying code of a graph $S \subseteq V(G)$ is an identifying restrained dominating set if $N[S] = V(G)$ and $\langle V(G) \setminus S \rangle$ is a subgraph without isolated vertices. The minimum cardinality of an identifying restrained dominating set of G, denoted by $\gamma_r^{ID}(G)$, is called the identifying restrained domination number of G. In this paper, we initiate the study of the concept and give the domination number of some special graphs. Further, we show the characterization of the identifying restrained dominating set in the corona of two nontrivial connected graphs.

Keywords: dominating set, identifying code, restrained dominating set, identifying restrained dominating set

1. Introduction

Domination in graph theory was introduced by Claude Berge in 1958 and Oystein Ore in 1962 [1]. Following an article [2] by Ernie Cockayne and Stephen Hedetniemi in 1977, the domination in graphs became an area of study by many researchers. A subset S of $V(G)$ is a *dominating set* of G if for every $v \in V(G) \setminus S$, there exists $x \in S$ such that $xv \in E(G)$, i.e. $N[S] = V(G)$. The *domination number* $\gamma(G)$ of G is the smallest cardinality of a dominating set of G. Some studies on domination in graphs were found in the papers [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17].

The identifying code of a graph was studied in 1998 by M.G. Karpovsky et al. [18] in their paper "On a new class of codes for identifying vertices in graphs." They observed that the concept of identifying codes is that a graph is identifiable if and only if it is twin-free. A vertex x is a twin of another vertex y if $N[x] = N[y]$. A graph G is called twin-free if no vertex has a twin. An identifying code of a graph G is a dominating set $C \subseteq V(G)$ such that for every $v \in V(G)$, $N_G[v] \cap C$ is distinct. The minimum cardinality of an identifying code of G, denoted by $\gamma^{ID}(G)$, is called the *identifying code number* of G. An identifying code of cardinality $\gamma^{ID}(G)$ is called an $\gamma^{ID}(G)$-set of G. From a computational point of view, it is shown that given a graph G, finding the exact value of $\gamma^{ID}(G)$ is in the class of NP-hard problems. It in fact remains NP-hard for many subclasses of graphs [19, 20]. Furthermore, approximating $\gamma^{ID}(G)$ is not easy as shown in [21, 22, 23]. Identifying code of a graph is also studied in [24].

The restrained domination in graphs was introduced by Telle and Proskurowski [26] indirectly as a vertex partitioning problem. Accordingly, a set $S \subseteq V(G)$ is a *restrained dominating set* if every vertex not in S is adjacent to a vertex in S and to a vertex in $V(G) \setminus S$. Alternately, a subset S of $V(G)$ is a restrained dominating set if $N[S] = V(G)$ and $\langle V(G) \setminus S \rangle$ is a subgraph without isolated vertices. The minimum cardinality of a restrained dominating set of G, denoted by $\gamma_r(G)$, is called the restrained domination number of G. A restrained dominating set of cardinality $\gamma_r(G)$ is called an $\gamma_r(G)$-set. Restrained domination in graphs was also found in the papers [27, 28, 29, 30, 31, 32, 33, 34, 35, 36].

The identifying code in graphs and restrained domination in graphs have motivated the researchers to introduce a new domination in graphs - an identifying restrained domination in graphs. An identifying code S of a graph G is an identifying restrained dominating set if every vertex not in S is adjacent to a vertex in S and to a vertex in $V(G) \setminus S$. Alternately, an identifying code of a graph $S \subseteq V(G)$ is an *identifying restrained dominating set* if $N[S] = V(G)$ and $\langle V(G) \setminus S \rangle$ is a subgraph without isolated vertices. The minimum cardinality of an identifying restrained dominating set of G, denoted by $\gamma_r^{ID}(G)$, is called the identifying restrained domination number of G. In this paper, we initiate the study of the concept and give the domination number of some special graphs. Further, we show the characterization of the identifying restrained dominating set in the corona of two nontrivial connected graphs.

For the general terminology in graph theory, readers may refer to [37]. A graph G is a pair $(V(G), E(G))$, where $V(G)$ is a finite nonempty set called the *vertex-set* of G and $E(G)$ is a set of unordered pairs $\{u, v\}$ (or simply uv) of distinct elements from $V(G)$ called the *edge-set* of G. The elements of $V(G)$ are called *vertices* and the cardinality $|V(G)|$ of $V(G)$ is the *order* of G. The elements of $E(G)$ are called *edges* and the cardinality $|E(G)|$ of $E(G)$ is the *size* of G. If $|V(G)| = 1$, then G is called a trivial graph. If $E(G) = \emptyset$, then G is called an empty graph. The *open neighborhood* of a vertex $v \in V(G)$ is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The elements of $N_G(v)$ are called *neighbors* of v. The closed neighborhood of $v \in V(G)$ is the set $N_G[v] = N_G(v) \cup \{v\}$. If $X \subseteq V(G)$, the *open neighborhood* of X in G is the set $N_G(X) = \bigcup_{v\in X} N_G(v)$. The *closed neighborhood* of X in G is the set $N_G[X] = \bigcup_{v\in X} N_G[v]$.

2. Results

Definition 2.1 [1] Let G be a directed or undirected graph with the vertex set V(G). A subset D of V(G) is a dominating set for G when every vertex not in D is the endpoint of some edge from a vertex in D. Clearly, V(G) itself is a dominating set. A minimum dominating set is a dominating set such that no subset has this property. The domination number$\gamma(G)$ of a graph G is the smallest number of vertices in any minimum dominating set.

Definition 2.2 [24] An identifying code of a graph G is a dominating set $C \subseteq V(G)$ such that for every vertex $v \in V(G)$, $N_G[v] \cap C$ is distinct. The minimum cardinality of an identifying code of G, denoted by $\gamma^{ID}(G)$, is called the identifying code number of G. An identifying code of cardinality $\gamma^{ID}(G)$ is called a γ^{ID}-set of G.

Definition 2.3 [39] Let $G = (V, E)$ be a graph. A restrained dominating set is a set $S \subseteq V$ where every vertex in $V - S$ is adjacent to a vertex in S as well as another vertex in $V - S$. Alternately, a subset S of V(G) is a restrained dominating set if $N[S] = V(G)$ and $\langle V(G) \setminus S \rangle$is a subgraph without isolated vertices. The restrained domination number of G, denoted by $\gamma_r(G)$, is the smallest cardinality of a restrained dominating set of G.

Remark 2.4 Every graph G has a restrained dominating set, since V(G) is such a set.

Definition 2.5 A dominating set S of vertices of a graph G is an identifying restrained dominating set of G if S is a restrained dominating set and for every two vertices x and y, the sets $N_G[x] \cap S$ and $N_G[y] \cap S$ are nonempty and distinct. The identifying restrained domination number of G, denoted by $\gamma_r^{ID}(G)$, is the minimum cardinality of an identifying restrained dominating set of G. An identifying restrained dominating set of cardinality$\gamma_r^{ID}(G)$ will be called γ_r^{ID}-set.

Remark 2.6 Let G be a non-complete graph. If $S \subseteq V(G)$ is both an identifying code and a restrained dominating set of G, then S is an identifying restrained dominating set of G.

Proposition 2.7 Let $G = C_n$ for all integers $n \geq 5$. Then

$$\gamma_r^{ID}(G) = \begin{cases} \frac{3n}{5}, & if\ n \equiv 0\ (mod\ 5), \\ \frac{3n+2}{5}, & if\ n \equiv 1\ (mod\ 5), \\ \frac{3n+4}{5}, & if\ n \equiv 2\ (mod\ 5), \\ \frac{3n+6}{5}, & if\ n \equiv 3\ (mod\ 5), \\ \frac{3n+8}{5}, & if\ n \equiv 4\ (mod\ 5). \end{cases}$$

Remark 2.8 If $G = C_n$, then V(G) is an identifying code of G for all $n \geq 4$.

We need the following results for our subsequent Theorem and Corollary.

Lemma 2.9 Let G be a connected non-complete graph and $H = C_n$ where $n = 2k + 4$ for all positive integer k. *If* $S = \bigcup_{v\in V(G)} S_H^v$*, where* $S_H^v \subseteq V(H)$ *is an identifying code of* H *for each* $v \in V(G)$*, then* S *is an identifying restrained dominating set of* $G \circ H$

Proof. Suppose that $S = \bigcup_{v\in V(G)} S_H^v$, where $S_H^v \subseteq V(H)$ is an identifying code of H for each $v \in V(G)$. Consider the following cases.

Case 1. If $S_H^v = V(H)$ for each $v \in V(G)$. Then $V(H)$ is an identifying code of H. Thus, $N_H[u] \cap V(H)$ is distinct for all $u \in V(H)$. Since $n = 2k + 4 > 4$ for all positive integer k, for each $v \in V(G)$ and $u \in V(H)$, $N_{G\circ H}[v] \cap V(H) \neq N_{G\circ H}[u] \cap V(H)$. This implies that for all $z \in V(G \circ H)$,

$$N_{G\circ H}[z] \cap \left(\bigcup_{v\in V(G)} V(H^v)\right) = N_{G\circ H}[z] \cap \left(\bigcup_{v\in V(G)} S_H^v\right) = N_{G\circ H}[z] \cap S$$

is distinct. Hence, S is an identifying code of $G \circ H$. Further,

$$\langle V(G \circ H) \setminus S\rangle = \langle V(G \circ H) \setminus \left(\bigcup_{v\in V(G)} V(H^v)\right)\rangle = \langle V(G)\rangle$$

is a subgraph of $G \circ H$ without isolated vertices. Thus, S is a restrained dominating set of $G \circ H$, that is, S is an identifying restrained dominating set of $G \circ H$.

Case 2. If $S_H^v \subset V(H)$ for each $v \in V(G)$. Let $S_H^v = \{u_{2k-1}: k = 1,2,3, \dots, \frac{n}{2}\}$ for each $v \in V(G)$. Then $N_H[u_{2k-1}] \cap S_H^v = \{u_{2k-1}\} \subset S_H^v$ and $N_H[u_{2k}] \cap S_H^v = \{u_{2k-1}, u_{2k+1}\} \subset S_H^v$ for all $k \in \{1, 2, 3, \dots, \frac{n}{2}\}$, that is, $N_H[u] \cap S_H^v \subset S_H^v$ is distinct for all $u \in V(H)$ and $v \in V(G)$. Now, for each $v \in V(G)$, $N_{G\circ H}[v] \cap S_H^v = S_H^v$. This implies that for each $v \in V(G)$ and $u \in V(H)$, $N_{G\circ H}[v] \cap S_H^v \neq N_{G\circ H}[u] \cap S_H^v$. Hence, for all $z \in V(G \circ H)$,

$$N_{G\circ H}[z] \cap \left(\bigcup_{v\in V(G)} S_H^v\right) = N_{G\circ H}[z] \cap S$$

is distinct. Hence, S is an identifying code of $G \circ H$. Further,

$$\langle V(G \circ H) \setminus S\rangle = \langle V(G \circ H) \setminus \left(\bigcup_{v\in V(G)} S_H^v\right)\rangle = \langle V(G) \cup \left(\bigcup_{v\in V(G)} (V(H^v) \setminus S_H^v)\right)\rangle$$

is a subgraph of $G \circ H$ without isolated vertices. Thus, S is a restrained dominating set of $G \circ H$, that is, S is an identifying restrained dominating set of $G \circ H$. ■

Lemma 2.10 Let G be a connected non-complete graph and $H = C_n$ where $n = 2k + 4$ for all positive integer k. *If* $S = \left(\bigcup_{v\in V(G)\setminus T} V(H^v)\right) \cup \left(\bigcup_{v\in T} S_H^v\right)$*, where* $\emptyset \neq T \subset V(G)$ *and* $\emptyset \neq S_H^v \subset V(H)$ *is an identifying code of* H*, then* S *is an identifying restrained dominating set of* $G \circ H$.

Proof. Suppose that $S = \left(\bigcup_{v\in V(G)\setminus T} V(H^v)\right) \cup \left(\bigcup_{v\in T} S_H^v\right)$, where $\emptyset \neq T \subset V(G)$ and $\emptyset \neq S_H^v \subset V(H)$ is an identifying code of H. Since $n = 2k + 4 > 4$ for all positive integer k, $V(H) = V(C_n)$ is an identifying code of H, by Remark 2.8. Note that for all positive integer i, if $i \neq 1$ and $i \neq n$, then $N_H[u_i] \cap V(H) = \{u_{i-1}, u_i, u_{i+1}\} \subset V(H)$, otherwise $N_H[u_1] \cap V(H) = \{u_n, u_1, u_2\} \subset V(H)$ and $N_H[u_n] \cap V(H) = \{u_{n-1}, u_n, u_1\} \subset V(H)$ since H is a cycle of order n. Further, for all $v \in V(G)$, then $N_{G\circ H}[v] \cap V(H) = V(H)$. This implies that for all $v \in V(G)$ and $u \in V(H)$, $N_{G\circ H}[v] \cap V(H) \neq N_{G\circ H}[u] \cap V(H)$, that is,

$$N_{G\circ H}[v] \cap \left(\bigcup_{v\in V(G)\setminus T} (V(H^v))\right) \neq N_{G\circ H}[u] \cap \left(\bigcup_{v\in V(G)\setminus T} (V(H^v))\right).$$

By Lemma 2.9, for all $z \in V(\langle T\rangle \circ H)$, $N_{\langle T\rangle\circ H}[z] \cap \left(\bigcup_{v\in T} S_H^v\right)$ is distinct. Thus,

$$N_{G\circ H}[v] \cap \left[\left(\bigcup_{v\in V(G)\setminus T} (V(H^v))\right) \cup \left(\bigcup_{v\in T} S_H^v\right)\right] \neq N_{G\circ H}[u] \cap \left[\left(\bigcup_{v\in V(G)\setminus T} (V(H^v))\right) \cup \left(\bigcup_{v\in T} S_H^v\right)\right],$$

that is, $N_{G\circ H}[v] \cap S \neq N_{G\circ H}[u] \cap S$. Hence, $S = \left(\bigcup_{v\in V(G)\setminus T} V(H^v)\right) \cup \left(\bigcup_{v\in T} S_H^v\right)$ is an identifying code of $G \circ H$. Further,

$$\langle V(G\circ H)\setminus S\rangle = \left\langle V(G\circ H)\setminus \left[\left(\bigcup_{v\in V(G)\setminus T}(V(H^v))\right)\cup\left(\bigcup_{v\in T}S_H^v\right)\right]\right\rangle$$
$$= \left\langle V(G)\cup\left(\bigcup_{v\in T}(V(H^v)\setminus S_H^v)\right)\right\rangle.$$

is a subgraph of $G\circ H$ without isolated vertices. Thus, S is a restrained dominating set of $G\circ H$, that is, S is an identifying restrained dominating set of $G\circ H$.

Definition 2.11 Let G and H be graphs of order m and n, respectively. The corona of two graphs G and H is the graph $G\circ H$ obtained by taking one copy of G and m copies of H, and then joining the ith vertex of G to every vertex of the ith copy of H. The join of vertex v of G and a copy H^v of H in the corona of G and H is denoted by $v + H^v$.

Theorem 2.12 Let G be a connected non-complete graph and $H = C_n$ where $n = 2k + 4$ for all positive integer k. The subset S is an identifying restrained dominating set of $G\circ H$, if S_H^v is an identifying code of H for each $v \in V(G)$ and one of the following conditions is satisfied.

(i) $S = \bigcup_{v\in V(G)} S_H^v$, *where* $S_H^v \subseteq V(H)$.

(ii) $S = \left(\bigcup_{v\in V(G)\setminus T} V(H^v)\right)\cup\left(\bigcup_{v\in T} S_H^v\right)$, *where* $\emptyset \neq T \subset V(G)$ *and* $\emptyset \neq S_H^v \subset V(H)$.

Proof. Suppose that statement *(i)* is satisfied. Then $S = \bigcup_{v\in V(G)} S_H^v$, where $S_H^v \subseteq V(H)$ is an identifying code of H for each $v \in V(G)$. By Lemma 2.9, S is an identifying restrained dominating set of $G\circ H$.

Suppose that statement *(ii)* is satisfied. Then $S = \left(\bigcup_{v\in V(G)\setminus T} V(H^v)\right)\cup\left(\bigcup_{v\in T} S_H^v\right)$, where $T \subset V(G)$ and $S_H^v \subset V(H)$. By Lemma 2.10, S is an identifying restrained dominating set of $G\circ H$.

The next result is an immediate consequence of Theorem 2.12.

Corollary 2.13 Let G be a connected non-complete graph of order m and $H = C_n$ where $n = 2k + 4$ for all positive integer k. Then $\gamma_r^{ID}(G\circ H) = \frac{mn}{2}$.

Proof. Suppose that G is connected non-complete graph of order m and $H = C_n$ with $V(H) = \{x_1, x_2, \dots, x_n\}$, $E(H) = \{x_1x_2, x_2x_3, \dots, x_{n-1}x_n, x_nx_1\}$, and $n = 2k + 4$ for all positive integer k. Let $S = \bigcup_{v\in V(G)} S_H^v$, where $S_H^v \subset V(H)$ is an identifying code of H. By Theorem 2.12,S is an identifying restrained dominating set of $G\circ H$. Thus,

$$\gamma_r^{ID}(G\circ H) \leq |S| = \left|\bigcup_{v\in V(G)} S_H^v\right| = \sum_{v\in V(G)} |S_H^v| = |V(G)|\cdot|S_H|$$

for all identifying code S_H. Let $S_H = \{x_1, x_3, \dots, x_{n-1}\}$. Then $|S_H| = \frac{(n-1)+1}{2} = \frac{n}{2}$. Suppose that S_H is not a minimum identifying code of H. Then there exists an identifying code of $S_H^{'}$ in H such that $|S_H^{'}| < |S_H|$. Let $x \in V(H)$. Consider the following cases.

Case 1. If $x \in S_H$, then $S_H^{'} = S_H \setminus \{x\} = \{x_1, x_3, \dots, x_{n-1}\} \setminus \{x\}$ is not a dominating set for any $x \in S_H$.

Case 2. If $x \notin S_H$, then $S_H^{'} = V(H) \setminus S_H = \{x_2, x_4, \dots, x_n\}$ is an identifying code of H, but $|S_H^{'}| = |S_H|$.

By following the two cases, we may conclude that S_H must be a minimum identifying code of H.

Since $|V(G)| = m$, it follows that $\gamma_r^{ID}(G\circ H) \leq |S| = |V(G)|\cdot|S_H| = \frac{mn}{2}$. Since S_H is a minimum identifying code of H, it is clear that $S = \bigcup_{v\in V(G)} S_H^v$, where S_H^v is a minimum identifying code of H^v for each $v \in V(G)$, must be a minimum identifying restrained dominating set of $G\circ H$. Therefore, $\gamma_r^{ID}(G\circ H) = |S| = |V(G)|\cdot |S_H| = \frac{mn}{2}$.■

3. Conclusion

In this work, we introduced a new parameter of domination on graphs - the identifying restrained domination of graphs. The identifying restrained domination in the corona of two graphs were characterized. The exact identifying restrained domination number resulting from this binary operation of two graphs were computed. This study will pave a way to new research such bounds and other binary operations of two graphs. Other parameters involving identifying restrained domination in graphs may also be explored. Finally, the characterization of an identifying restrained domination in graphs may be the subject of further study.

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