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Sum of Secondary Orthogonal Bimatrices in R_{nxn}

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Abstract: Let $F \in \{R, C, H\}$. Let $\boldsymbol{\mathcal{U}_{n\times n}}$ be the set of secondary unitary bimatrics in $F_{n\times n}$, and let $O_{n\times n}$ be the set of secondary orthogonal bimatrices in $F_{n\times n}$. Suppose $n \geq 2$, we show that every $A_B \in F_{n\times n}$ can be written as a sum of bimatrices in $\boldsymbol{\mathcal{U}_{n\times n}}$ and of bimatrices in $O_{n\times n}$. let $A_B \in F_{n\times n}$ be given that and let $k \geq 2$ be the least integer that is a least upper bound of the singular values of A_B. When F=R, we show that if $k \leq 3$, then A_B can be written as a sum of 6 secondary orthogonal bimatrices; if $k \geq 4$, we show that A_B can be written as a sum of $k + 2$ secondary orthogonal bimatrices.

Keywords: Orthogonal matrix, unitary matrix, bimatrix, orthogonal bimatrix, unitary bimatrix, secondary orthogonal bimatrices, secondary unitary bimatrices.

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I. Introduction

Matrices provide a very powerful tool for dealing with linear models. Bimatrices are still a powerful and an advanced tool which can handle over one linear model at a time. Bimatrices are useful when time bound comparisons are needed in the analysis of a model. Bimatrices are of several types. We denote the space of nxn complex matrices by $\mathcal{R}_{\text{nxn}}$. For$A \in C_{\text{nxn}}$, A^T, A^s, A^*, A^{-1} and $det(A)$ denote transpose, secondary transpose, conjugate transpose, inverse and determinant of A respectively. If $AA^T = A^T A = I$ then A is an orthogonal matrix, where Iis the identity matrix.If $AVA^T V = VA^T V\ A = I\ or\ AA^S = A^S A = I,$ Where V is a permutation matrix with units in its secondary diagonal, A^S is a secondary orthogonal matrix. In this paper, we study secondary orthogonal bimatrices as a generalization of secondary orthogonal matrices. Some of the properties of secondary orthogonal matrices are extended to secondary orthogonal bimatrices. Some important results of secondary orthogonal matrices are generalized to secondary orthogonal bimatrices.

II. Basic Definitions and Results

Definition 2.1 [1]

A bimatrix A_B is defined as the union of two rectangular array of numbers A_1 and A_2 arranged into rows and columns. It is written as $A_B = A_1 \cup A_2$ with $A_1 \neq A_2$ (except zero and unit bi matrices) where,

$$A_1 = \begin{bmatrix} a_{11}^1 & a_{12}^1 & \cdots & a_{1n}^1 \\ a_{21}^1 & a_{22}^1 & \cdots & a_{2n}^1 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}^1 & a_{m2}^1 & \cdots & a_{mn}^1 \end{bmatrix} \text{and } A_2 = \begin{bmatrix} a_{11}^2 & a_{12}^2 & \cdots & a_{1n}^2 \\ a_{21}^2 & a_{22}^2 & \cdots & a_{2n}^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}^2 & a_{m2}^2 & \cdots & a_{mn}^2 \end{bmatrix}$$

$'\cup'$ is just for the notational convenience (symbol) only.

Definition 2.2 [1]

Let $A_B = A_1 \cup A_2$ and $C_B = C_1 \cup C_2$ be any two mx n bimatrices. The sum D_B of the bimatrices A_B and C_B is defined as

$$\begin{aligned} D_B = A_B + C_B &= (A_1 \cup A_2) + (C_1 \cup C_2) \\ &= (A_1 + C_1) \cup (A_2 + C_2) \end{aligned}$$

Where $A_1 + C_1$ and $A_2 + C_2$ are the usual addition of matrices.

Definition 2.3 [2]

If $A_B = A_1 \cup A_2$ and $C_B = C_1 \cup C_2$ be two bimatrices, then A_B and C_B are said to be equal (written as $A_B = C_B$) if and only if A_1 and C_1 are identical and A_2 and C_2 are identical. (That is, A_1= C_1 and A_2 = C_2).

Definition 2.4 [2]

Given a bimatrix $A_B = A_1 \cup A_2$ and a scalar λ, the product of λ and A_B written as λA_B is defined to be

$$\lambda A_B = \begin{bmatrix} \lambda a_{11}^1 & \lambda a_{12}^1 & \cdots & \lambda a_{1n}^1 \\ \lambda a_{21}^1 & \lambda a_{22}^1 & \cdots & \lambda a_{2n}^1 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{m1}^1 & \lambda a_{m2}^1 & \cdots & \lambda a_{mn}^1 \end{bmatrix} \cup \begin{bmatrix} \lambda a_{11}^2 & \lambda a_{12}^2 & \cdots & \lambda a_{1n}^2 \\ \lambda a_{21}^2 & \lambda a_{22}^2 & \cdots & \lambda a_{2n}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{m1}^2 & \lambda a_{m2}^2 & \cdots & \lambda a_{mn}^2 \end{bmatrix} = (\lambda A_1 \cup \lambda A_2).$$

That is, each element of A_1 and A_2 are multiplied by λ.

Remark 2.5 [2]

If $A_B = A_1 \cup A_2$be a bimatrix, then we call A_1 and A_2 as the component matrices of the bimatrix A_B.

Definition 2.6 [1]

If $A_B = A_1 \cup A_2$ and $C_B = C_1 \cup C_2$ are both *n*x *n* square bimatrices then, the bimatrix multiplication is defined as, $A_B \times C_B = (A_1C_1) \cup (A_2C_2)$.

Definition 2.7 [1]

Let $A_B^{m\times m} = A_1 \cup A_2$be a *m*x*m* square bimatrix. We define $I_B^{m\times m} = I^{m\times m} \cup I^{m\times m} = I_1^{m\times m} \cup I_2^{m\times m}$ to be the identity bimatrix.

Definition 2.8 [1]

Let $A_B^{m\times m} = A_1 \cup A_2$ be a square bimatrix, A_B is a symmetric bimatrix if the component matrices A_1 and A_2 are symmetric matrices. i.e, $A_1 = A_1^T$ and $A_2 = A_2^T$.

Definition 2.9 [1]

Let $A_B^{m\times m} = A_1 \cup A_2$ be a *m*x*m* square bimatrix i.e, A_1 and A_2 are *m*x*m* square matrices. A skew-symmetric bimatrix is a bimatrix A_B for which $A_B = -A_B^T$, where $-A_B^T = -A_1^T \cup -A_2^T$ i.e, the component matrices A_1 and A_2 are skew-symmetric.

Definition 2.10 [3]

A bimatrix $A_B = A_1 \cup A_2$ is said to be unitary bimatrix, if $A_BA_B^* = A_B^*A_B = I_B$ (or) $(A_1A_1^* \cup$ $A2A2*=A1*A1 \cup A2*A2=I1 \cup I2.$
(That is, the component matrices of A_B are unitary.)

That is, $A_B^* = A_B^{-1}$ (or) $(A_1^* \cup A_2^*) = (A_1^{-1} \cup A_2^{-1})$.

Definition 2.11 [4]

A bimatrix $A_B = A_1 \cup A_2$ is said to be orthogonal bimatrix, if $A_BA_B^T = A_B^TA_B = I_B$ (or) $(A_1A_1^T \cup$ $A2A2T=A1TA1 \cup A2TA2=I1 \cup I2.$
(That is, the component matrices of A_B are orthogonal.)

That is, $A_B^T = A_B^{-1}$ (or) $(A_1^T \cup A_2^T) = (A_1^{-1} \cup A_2^{-1})$.

III. Secondary Orthogonal and Secondary Unitary Bimatrices

Definition 3.1 [5]

A bimatrix $A_B = A_1 \cup A_2$ is said to be secondary orthogonal bimatrix, if $A_BV_BA_B^TV_B = V_BA_B^TV_BA_B = I_B$ or$A_BA_B^S = A_B^SA_B = I_B$, where V_B is a permutation bimatrix with units in its secondary diagonal.
(That is, the component matrices of A_B are secondary orthogonal.)

That is, $A_B^S = A_B^{-1}$ (or) $(A_1^S \cup A_2^S) = (A_1^{-1} \cup A_2^{-1})$.

Remark 3.2

Let $A_B = A_1 \cup A_2$ be a secondary orthogonal bimatrix. If A_1 and A_2 are square and possess the same order then A_B is called square secondary orthogonal bimatrix, and if A_1 and A_2 are of different orders then A_B is called mixed square secondary orthogonal bimatrix.

Example 3.3

(1) $A_B = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cup \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$ is a square secondary orthogonal bimatrix.

(2) $A_B = \begin{bmatrix} cos\,\theta & sin\,\theta \\ -sin\,\theta & cos\,\theta \end{bmatrix} \cup \begin{bmatrix} cos\,\theta & -sin\,\theta & 0 \\ sin\,\theta & cos\,\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a mixed square secondary orthogonal bimatrix.

Definition 3.4 [4]

Let $A_B = A_1 \cup A_2$ be an $n \times n$ complex bimatrix. (A bimatrix A_B is said to be complex if it takes entries from the complex field). A_B is called a unitary bimatrix if $A_B A_B^* = A_B^* A_B = I_B$ (or) $\bar{A}_B^{\,T} = A_B^{-1}$.

That is, $A_1 A_1^* \cup A_2 A_2^* = A_1^* A_1 \cup A_2^* A_2 = I_1 \cup I_2$.

Example 3.5

$A_B = A_1 \cup A_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} i & i \\ i & -i \end{bmatrix} \cup \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$ is a unitary bimatrix.

In this paper, we have determined which bimatrices (if any) in $R_{n\times n}$can be written as a sum of secondary unitary or secondary orthogonal bimatrices. Also, we have obtained that if $k \leq 3$, thenA_B can be written as a sum of six secondary orthogonal bimatrices, and if $k \geq 4$, thenA_B can be written as a sum of $k+2$ secondaryorthogonal bimatrices, where *k* be the least integer that is a least upper bound of the singular values of A_B. We let $\boldsymbol{\mathcal{U}_{n\times n}}$ and $O_{n\times n}$ are the set of secondary unitary and secondary orthogonal bimatrices in the complex field. We begin with the following observation.

Lemma 3.6

Let *n* be a given positive integer. Let $G \subset F_{n\times n}$ be a group under multiplication. Then $A_B \in F_{n\times n}$can be written as a sum of bimatrices in *G* if and only if for every $Q_B, P_B \in G$, the bimatrix $Q_B A_B P_B$ can be written as a sum of bimatrices in *G*.

Notice that both $\boldsymbol{\mathcal{U}_{n\times n}}$ and $O_{n\times n}$ are groups under multiplication.

Let $\alpha_1, \alpha_2 \in F$ be given. Then lemma 3.6 guarantees that for each $Q_B \in G$, we have that $\alpha_1 Q_1 \cup \alpha_2 Q_2$ can be written as a sum of bimatrices from *G* if and only if $\alpha_1 I_1 \cup \alpha_2 I_2$ can be written as a sum of bimatrices from *G*.

Lemma 3.7

Let $n \geq 2$ be a given integer. Let $G \subset F_{n\times n}$ be a group under multiplication. Suppose that *G* contains $K_B \equiv diag(1, -1, \ldots, -1)$ and the permutation bimatrices. Then every $A_B \in F_{n\times n}$ can be written as a sum of bimatrices in *G* if and only if for each $\alpha_1, \alpha_2 \in F, \alpha_1 I_1 \cup \alpha_2 I_2$ can be written as a sum of bimatrices from *G*.

IV. Sum of Secondary Orthogonal Bimatrices in R_{nxn}

The only bimatrices in the set of all secondary orthogonal bimatrices of order 1 are ±1. Hence, not every element of $F_{1\times 1}$ can be written as a sum of elements in the set of all secondary orthogonal bimatrices of order 1. In fact, only the integers can be written as a sum of elements of the set of all secondary orthogonal bimatrices of order 1.

Notice that $O_n(\mathbb{R}) = u_n(\mathbb{R})$. When *n*=1, only the integers can be written as a sum of elements of $O_1(\mathbb{R})$. Suppose that *n*=2. We mimic the computations done in the case when $F = \mathbb{R}$.

Let $\theta_1, \theta_2 \in \mathbb{R}$be given, set $\alpha_1 = Cos\,\theta_1;\ \alpha_2 = Cos\,\theta_2$ and set $\beta_1 = Sin\,\theta_1;\ \beta_2 = Sin\,\theta_2$

Then $[A_1(\alpha_1, \beta_1) \cup A_2(\alpha_2, \beta_2)]$in equation (2) of **[6]** is an element of $O_2(\mathbb{R})$.

Moreover,$[(A_1^{\mathrm{i}} + A_1^{\mathrm{ii}}) \cup (A_2^{\mathrm{i}} + A_2^{\mathrm{ii}})] = 2[Cos\theta_1 I_1^{\mathrm{i}} \cup Cos\theta_2 I_2^{\mathrm{ii}}]$.

Now, for every $\delta_1, \delta_2 \in \mathbb{R}$there exist a positive integer m and $\theta_1, \theta_2 \in \mathbb{R}$such that $2m\,Cos\,\theta_1 = \delta_1;\ 2m\,Cos\,\theta_2 = \delta_2$.

We conclude that every $(A_1 \cup A_2) \in \mathbb{R}_{n\times n}$can be written as a sum of an even number of bimatrices from $O_2(\mathbb{R})$.

When *n*=3, we again mimic the computations done in the case when $F = \mathbb{C}$using $\alpha_1 = Cos\,\theta_1;\ \alpha_2 = Cos\,\theta_2$and $\beta_1 = Sin\,\theta_1;\ \beta_2 = Sin\,\theta_2$to show that for every $\delta_1, \delta_2 \in \mathbb{R}$the bimatrix $(\delta_1 I_1^{\mathrm{iii}} \cup \delta_2 I_2^{\mathrm{iii}})$can be written as a sum of an even number of bimatrices from $O_3(\mathbb{R})$.

Let $n \geq 4$be a given integer. If *n*=2*k* is even, then write $(\delta_1 I_1^{2k} \cup \delta_2 I_2^{2k}) = (\delta_1 I_1^{\mathrm{ii}} \cup \delta_2 I_2^{\mathrm{ii}}) \oplus \ldots \oplus (\delta_1 I_1^{\mathrm{ii}} \cup \delta_2 I_2^{\mathrm{ii}})$ (k copies), and note that each $(\delta_1 I_1^{\mathrm{ii}} \cup \delta_2 I_2^{\mathrm{ii}})$can be written as a sum of an even number of bimatrices from $O_2(\mathbb{R})$.

If *n*=2*k*+1 is odd, then write $(\delta_1 I_1^{2k+1} \cup \delta_2 I_2^{2k+1}) = (\delta_1 I_1^{2n-2} \cup \delta_2 I_2^{2n-2}) \oplus (\delta_1 I_1^{\mathrm{iii}} \cup \delta_2 I_2^{\mathrm{iii}})$.

Now, $(\delta_1 I_1^{2n-2} \cup \delta_2 I_2^{2n-2})$ can be written as a sum of an even number of bimatrices from $O_{2n-2}(\mathbb{R})$and $(\delta_1 I_1^{\mathrm{iii}} \cup \delta_2 I_2^{\mathrm{iii}})$ can be written as a sum of an even number of matrices from $O_{2n-2}(\mathbb{R})$and $(\delta_1 I_1^{\mathrm{iii}} \cup \delta_2 I_2^{\mathrm{iii}})$ can be written as a sum of an even number of bimatrices from $O_3(\mathbb{R})$. We conclude that $(\delta_1 I_1^{2k+1} \cup \delta_2 I_2^{2k+1})$can be written as a sum of an even number of bimatrices from $O_{2k+1}(\mathbb{R})$.

Hence, Lemma 3.2 of **[6]** guarantees that for every integer $n \geq 2$, every $(A_1 \cup A_2) \in \mathbb{R}_{n\times n}$ can be written as a sum of bimatrices from $O_n(\mathbb{R})$.

Theorem 4.1

Let $n \geq 2$ be a given integer. Every $(A_1 \cup A_2) \in \mathbb{R}_{n\times n}$ can be written as a sum of bimatrices from $\boldsymbol{O_n}(\mathbb{R}) = \boldsymbol{\mathcal{U}_n}(\mathbb{R})$

Proof

Let $n \geq 2$ be a given integer and let $(U_1 \cup U_2) \in \boldsymbol{\mathcal{U}_n}(\mathbb{R})$ be given.

Then $(U_1 \cup U_2) \in \boldsymbol{\mathcal{U}_n}(\mathbb{R}) \cap \mathbf{O_n}(\mathbb{R})$ that is, a realsecondary orthogonal bimatrix is both complex secondary unitary bimatrix and complexsecondary orthogonal bimatrix.

Hence,$(A_1 \cup A_2) \in \mathbb{R}_{n\times n}$which a sum of matrices is in $\boldsymbol{\mathcal{U}_n}(\mathbb{R})$ is both a sum of complex secondary unitary bimatrices and a sum of complex secondary orthogonal bimatrices. Thus, the restrictions on these cases apply. It k is a positive integer such that$\sigma_1^1(A_1) > k;\ \sigma_2^1(A_2) > k$,then $(A_1 \cup A_2)$cannot be written as a sum of k real secondary orthogonal bimatrices.

Let m be a positive integer. Theorem 3.9 of **[6]** guarantees that $(I_1 \cup I_2) \in \mathbb{C}_{2m+1}$ cannot be written as a sum of two bimatrices in $O_{2m+1}(\mathbb{R})$**.**

Now, we cannot be written as a sum of two bimatrices from $O_{2m+1}(\mathbb{R}) \subset O_{2m+1}(\mathbb{R})$.

In general, if $\alpha_1, \alpha_2 \notin \{-2,0,2\}$and if $(Q_1 \cup Q_2) \in O_{2m+1}(\mathbb{R})$, then $(\alpha_1 Q_1 \cup \alpha_2 Q_2)$ cannot be written as a sum of two bimatrices from $O_{2m+1}(\mathbb{R})$.

Let $n \geq 2$ be a given integer, and let $(A_1 \cup A_2) \in \mathbb{R}_{n\times n}$ be given. We now look at the bimatrices in $O_n(\mathbb{R})$ that make up the sum $(A_1 \cup A_2)$.

Definition 4.2

Let $\theta_1, \theta_2 \in \mathbb{R}$ be given. We define

$$[A_1(\theta_1) \cup A_2(\theta_2)] \equiv \begin{bmatrix} Cos\ \theta_1 & Sin\ \theta_1 \\ -Sin\ \theta_1 & Cos\ \theta_1 \end{bmatrix} \cup \begin{bmatrix} Cos\ \theta_2 & Sin\ \theta_2 \\ -Sin\ \theta_2 & Cos\ \theta_2 \end{bmatrix} \text{ and}$$

$$[B_1(\theta_1) \cup B_2(\theta_2)] \equiv \begin{bmatrix} Cos\ \theta_1 & Sin\ \theta_1 \\ Sin\ \theta_1 & Cos\ \theta_1 \end{bmatrix} \cup \begin{bmatrix} Cos\ \theta_2 & Sin\ \theta_2 \\ Sin\ \theta_2 & Cos\ \theta_2 \end{bmatrix} \quad (1)$$

Remark 4.3

Set $(K_1^n \cup K_2^n) \equiv [B_1(0) \cup B_2(0)]$and notice that $[A_1(0) \cup A_2(0)] = (I_1^n \cup I_2^n)$.

Let $0 \leq r, s \in \mathbb{R}$be given, and let $k \geq 2$be an integer. If $r, s \leq k$,then Lemma 3.1 of **[6]** and taking the real and imaginary parts of the equation $e^{i\theta_1^1} + \cdots + e^{i\theta_k^1} = \alpha_1;\ e^{i\theta_1^2} + \cdots + e^{i\theta_k^2} = \alpha_2$ (2)

Show that there exist $(\theta_1^1, \theta_2^1, \ldots, \theta_k^1) \in \mathbb{R};\ (\theta_1^2, \theta_2^2, \ldots, \theta_k^2) \in \mathbb{R}$ such that $[A_1(\theta_1^1) + \cdots + A_1(\theta_k^1)] \cup [A_2(\theta_1^2) + \cdots + A_2(\theta_k^2)] = r[I_1^n \cup I_2^n]$.

Moreover, there exist $(\beta_1^1, \ldots, \beta_k^1) \in \mathbb{R};\ (\beta_1^2, \ldots, \beta_k^2) \in \mathbb{R}$ such that $[B_1(\beta_1^1) + \cdots + B_1(\beta_k^1)] \cup [B_2(\beta_1^2) + \cdots + B_2(\beta_k^2)] = S[K_1^n \cup K_2^n]$

Theorem 4.4

Let a positive integer n and let $(A_1 \cup A_2) \in \mathbb{R}_{2n}$ be given. Suppose that $k \geq 2$is an integer such that $\sigma_1^1(A_1) \leq k;\ \sigma_2^1(A_2) \leq k$. Then $(A_1 \cup A_2)$ can be written as a sum of $2k$ matrices in $O_{2n}(\mathbb{R})$. Moreover, for every integer $m \geq 2k$ the matrix $(A_1 \cup A_2)$ can be written as a sum of m matrices in $O_{2n}(\mathbb{R})$.

Proof

Let $(A_1 \cup A_2) = (U_1 \cup U_2)(\Sigma_1 \cup \Sigma_2)(W_1 \cup W_2)$ be a singular value decomposition of $(A_1 \cup A_2)$.

Then Lemma 3.6 guarantees that we only need to concern ourselves with ε. For n=1, notice that $diag_B(\sigma_1^1, \sigma_1^2) \cup diag_B(\sigma_2^1, \sigma_2^2) = s[I_1^n \cup I_2^n] + r[K_1^n \cup k_2^n]$, where $s = \frac{1}{2}(\sigma_1^1 + \sigma_1^2) = \frac{1}{2}(\sigma_2^1 + \sigma_2^2)$ and $t = \frac{1}{2}(\sigma_1^1 - \sigma_1^2) = \frac{1}{2}(\sigma_2^1 - \sigma_2^2)$.

Now, $0 \leq t \leq s \leq k$. Hence, $s(I_1^n \cup I_2^n)$ and $t(K_1^n \cup k_2^n)$ can each be written as a sum of k secondary orthogonal bimatices. Moreover, for each integer $p \geq k$, notice that $s(I_1^n \cup I_2^n)$ can be written as a sum of p secondary orthogonal bimatrices.

Hence, $[(sI_1^n + rK_1^n) \cup (sI_2^n + rK_2^n)]$ can be written as a sum of p+k secondary orthogonal bimatrices.

For, $n > 1$write

$(\Sigma_1 \cup \Sigma_2) = diag(\sigma_1^1, \sigma_2^1, \dots, \sigma_{2n-1}^1, \sigma_{2n}^1) \cup diag(\sigma_1^2, \sigma_2^2, \dots, \sigma_{2n-1}^2, \sigma_{2n}^2)$
$= \left(diag(\sigma_1^1, \sigma_2^1) \oplus \dots \oplus diag(\sigma_{2n-1}^2, \sigma_{2n}^1)\right) \cup \left(diag(\sigma_1^2, \sigma_2^2) \oplus \dots \oplus diag(\sigma_{2n-1}^2, \sigma_{2n}^2)\right)$

Notice now that for each $j = 1, \dots, n, diag\left(\sigma_{2j-1}^1, \sigma_{2j}^1\right) \cup diag\left(\sigma_{2j-1}^2, \sigma_{2j}^2\right)$ can be written as a fun of $2k$ secondaryorthogonal bimatrices, say $\left(P_{j1}^1 \cup P_{j1}^2\right), \dots, \left(P_{j(2k)}^1 \cup P_{j(2k)}^2\right)$

For each $l = 1, \dots, 2k$,set$(Q_l^1 \cup Q_l^2) \equiv (P_{1l}^1 \cup P_{1l}^2) \oplus \dots \oplus (P_{nl}^1 \cup P_{nl}^2)$,and notice that$\Sigma = (Q_1^1 + \cdots +$ *Q2k1∪Q12+…+Q2k2*

Finally, notice that for each integer $m \geq 2k$ and for each$j = 1, \dots, n$, the matrix $diag\left(\sigma_{2j-1}^1, \sigma_{2j}^1\right) \cup diag\left(\sigma_{2j-1}^2, \sigma_{2j}^2\right)$ can be written as a sum of *m* secondary orthogonal bimatrices.

Remark 4.5

Consider $(C_0^1 \cup C_0^2) \equiv [diag(b_1, a_1) \cup diag(b_2, a_2)]$ with real numbers $b_1, b_2 \geq a_1, a_2 \geq 0$.

If $b_1, b_2 \geq 2$,then Theorem 3.4 ensures that $(C_0^1 \cup C_0^2)$ can be written as a sum of 4 real secondary orthogonal bimatrices. Moreover, for each integer$t \geq 4, (C_0^1 \cup C_0^2)$ can be written as a sum of t real secondary orthogonal bimatrices.

Suppose that $b_1, b_2 \leq 3$ if $0 \leq b_1 \leq 2;\ 0 \leq b_1 \leq 2$,then Theorem 3.4 guarantees that $(C_0^1 \cup C_0^2)$ can be written as a sum of four reals econdary orthogonal bimatrices. Moreover, $(C_0^1 \cup C_0^2)$ can also be written as a sum of five real secondary orthogonal bimatrices.

If $2 < b_1 \leq 3;\ 2 < b_2 \leq 3$,then we look at two cases:

(i) $0 \leq a_1 \leq 1;\ 0 \leq a_2 \leq 1$ and

(ii) $1 \leq a_1 \leq 3;\ 1 \leq a_2 \leq 3$

In the first case, set $(C_1^1 \cup C_2^1) \equiv (C_1^0 \cup C_2^0) - (K_1^2 \cup K_2^2)$. Then $0 \leq b_1 - 1 \leq 2;\ 0 \leq b_2 - 1 \leq 2$ and $0 \leq a_1 + 1 < 2;\ 0 \leq a_2 + 1 < 2$. Notice now that for each integer $t \geq 4, (C_1^1 \cup C_2^1)$ can be written as a sum of *t* real secondary orthogonal bimatrices.

In the second case, set $(C_1^1 \cup C_2^1) \equiv (C_1^0 - I_1^{\mathrm{n}}) \cup (C_2^0 - I_2^{\mathrm{n}})$. Then we have $0 \leq a_1 - 1 \leq b_1 - 1 \leq 2; 0 \leq a_2 - 1 \leq b_2 - 1 \leq 2$. Theorem 3.4 guarantees that for each integer $t \geq 4, (C_1^1 \cup C_2^1)$ can be written as a sum of t real secondary orthogonal bimatrices. Hence, for each integer $t \geq 5, (C_1^0 \cup C_2^0)$ can be written as a sum of t real secondary orthogonal bimatrices.

We now use induction to show that if $k \geq 2$is an integer satisfying $b_1 \leq k;\ b_2 \leq k$,then for each integer $t \geq k + 2, (C_1^0 \cup C_2^0)$can be written as a sum of t real secondary orthogonal bimatrices.

Suppose that the claim is true for some integer $k \geq 3$. We show that the claim is true when $0 < b_1 \leq k + 1;\ 0 < b_2 \leq k + 1$. if $0 \leq b_1 \leq k;\ 0 \leq b_2 \leq k$, then our inductive hypothesis guarantees that for each integer $t \geq k + 2, (C_1^0 \cup C_2^0)$ can be written as a sum of$t$ and hence, also of $t \geq k + 3$real secondary orthogonal bimatrices.

If $k < b_1 \leq k + 1;\ k < b_2 \leq k + 1$,we take a look at two cases:

(i) $1 \leq a_1 \leq k + 1;\ 1 \leq a_2 \leq k + 1$ and (ii) $0 \leq a_1 \leq 1;\ 0 \leq a_2 \leq 1$;

In case (i), set $(C_1^1 \cup C_2^1) \equiv (C_1^0 \cup C_2^0) - (I_1^{\mathrm{n}} \cup I_2^{\mathrm{n}})$; and in case (ii),
set $(C_1^1 \cup C_2^1) \equiv (C_1^0 \cup C_2^0) - (K_1^{\mathrm{n}} \cup K_2^{\mathrm{n}})$.

Lemma 4.6

Let $(C_1 \cup C_2) \in M_2(\mathbb{R})$ be given suppose that $k \geq 2$ is an integer such that $\sigma_1^1(C_1) \leq k$ and $\sigma_2^1(C_2) \leq k$. Then for each integer $t \geq k + 2, (C_1 \cup C_2)$ can be written as a sum of *t* matrices from $u_2(\mathbb{R})$.

Let $(A_1 \cup A_2) \in \mathbb{R}_{2n}$ be given, and suppose that the bi singular values of $(A_1 \cup A_2)$ are $\sigma_1^1 \geq \cdots \geq \sigma_1^{2n} \geq 0; \sigma_2^1 \geq \cdots \geq \sigma_2^{2n} \geq 0$.

Set $(D_1 \cup D_2) \equiv [diag(\sigma_1^1, \dots, \sigma_1^{2n}) \cup diag(\sigma_2^1, \dots, \sigma_2^{2n})]$

Write $(D_1 \cup D_2) \equiv [diag(\sigma_1^1, \dots, \sigma_1^2) \oplus \dots \oplus diag(\sigma_1^{2n-1}, \sigma_1^{2n})]$
$\cup \left(diag(\sigma_2^1, \dots, \sigma_2^2) \oplus \dots \oplus diag(\sigma_2^{2n-1}, \sigma_2^{2n})\right)]$.

Let $k \geq 2$ be an integer such that $\sigma_1^1(A) \leq k; \sigma_2^1(A_2) \leq k$. Then Lemma 4.6 guarantees that for each integer $t \geq k + 2$, and for each $j = 1, \dots, n, diag\left(\sigma_1^{2j-1}, \sigma_1^j\right) \cup diag\left(\sigma_2^{2j-1}, \sigma_2^j\right)$, can be written as a sum of *t* real secondary orthogonal bimatrices. We conclude that for each integer $t \geq k + 2, (A_1 \cup A_2)$ can be written as a sum of *t* real secondary orthogonal bimatrices.

Theorem 4.7

Let n be a positive integer, and let $(A_1 \cup A_2) \in \mathbb{R}_{2n}$be given. Suppose that $k \geq 2$is an integer such that $\sigma_1^1(A_1) \leq k$; $\sigma_2^1(A_2) \leq k$. then for each integer $t \geq k+2$, $(A_1 \cup A_2)$can be written as a sum of t matrices in $u_{2n}(\mathbb{R})$.

Proof

Let $(A_1 \cup A_2) \in \mathbb{R}_{3\times3}$be given.

Suppose that $(A_1 \cup A_2) = (P_1 \cup P_2)(\Sigma_1 \cup \Sigma_2)(Q_1 \cup Q_2)$, with $(P_1 \cup P_2), (Q_1 \cup Q_2) \in O_3(\mathbb{R})$ and $(\Sigma_1 \cup \Sigma_2) = [diag(a_1, b_1, c_1) \cup diag(a_2, b_2, c_2)]$ with $0 \leq c_1 \leq b_1 \leq a_1 \leq 2; 0 \leq c_2 \leq b_2 \leq a_2 \leq 2$.

If $a_1 = a_2 = 2$,then notice that $\big(diag(b_1, c_1) \cup diag(b_2, c_2)\big)$ can be written as a sum of four secondary orthogonal bimatrices. One checks that $(\Sigma_1 \cup \Sigma_2)$ can be written as a sum of four real secondary orthogonal bimatrices.

Suppose $a_1 < 2$; $a_2 < 2$. if $c_1 = c_2 = 0$,then $(\Sigma_1 \cup \Sigma_2)$ can be written as a sum of four secondary orthogonal bimatrices. If $c_1 = c_2 = 2$, then $(A_1 \cup A_2)$ is a sum of two secondary orthogonal bimatrices. If $0 \neq c_1 < 2$; $0 \neq c_2 < 2$, then, choose θ_1, θ_2 that $2\,Cos\,\theta_1 = c_1; 2\,Cos\,\theta_2 = c_2$.

Notice that $\big[\big(A_1(\theta_1) + A_1(-\theta_1)\big) \cup \big(A_2(\theta_2) + A_2(-\theta_2)\big)\big] = 2[Cos\,\theta_1 I_1^{\prime\prime} \cup Cos\,\theta_2 I_2^{\prime\prime}]$

Set $(U_1^{\prime} \cup U_2^{\prime}) = \big([1] \oplus A_1(\theta_1)\big) \cup \big([1] \oplus A_2(\theta_2)\big)$ and

set $(U_1^{\prime\prime} \cup U_2^{\prime\prime}) = \big([-1] \oplus A_1(-\theta_1)\big) \cup \big([-1] \oplus A_2(-\theta_2)\big)$.

Then $(\Sigma_1 \cup \Sigma_2) - \big((U_1^{\prime} \cup U_1^{\prime\prime}) + (U_2^{\prime} \cup U_2^{\prime\prime})\big) = \big(diag(a_1, b_1 - c_1, 0) \cup diag(a_2, b_2 - c_2, 0)\big)$, which can be written as a sum of four real secondary orthogonal bimatrices. Hence, $(A_1 \cup A_2)$ can be written as a sum of six real secondary orthogonal bimatrices.

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