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Approximate Solution of Real Definite Integration

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Abstract: In this study we present a mixed quadrature rule of degree of precision eleven by suitable convex combination of different Newton's open type formula of lower precision, by removing end point singularity for the product integral on the improper integral $I(f) = \int_0^\infty w(x) f(x) dx$,where $w(x)$ is the weight function and $f(x)$ is smooth function. This result has compared with mixed method for the product integral for approximate solution of real definite integrals on the infinite interval by mixing cubic Newton' divided difference and Romberg method. An error bound has been obtained by mixed quadrature on different classes of functions. Numerical examples have presented to validate and compared our present method with mixed methods.
Keywords: Product integral, Mixed quadrature, degree of precision, Maclaurin's expansion, Error bound.
MSC 2010: 65D30, 65D32.

1. Introduction

In this section an introduction to mixed quadrature rule and product integral has been mentioned.

There are several rules for evaluating the real definite integral of the form $I(f) = \int_a^b f(x) dx$ (1)

Here Eq. (1) can be approximated by different Newton-cotes -open type rules with the help of monomial transformation $2x = (b-a)t + (b+a)$ (2) to transform $x \in [a,b]$ to $x \in [-1, 1]$. There are many techniques on reducing the infinite interval into finite integral, but reduction technique brings singularity of the integrand functions and application of Romberg rule is not suitable for those parts where singularity appears. Again to avoid the singularity on product integral an attempt has taken to find approximate solution by mixing Newton's divided difference formula and Romberg method(Z. K. Eshkuvatov etal [10]) by transforming infinite interval $x \in [0, \infty]$ to $t \in [0,1]$. In present case, the double transformation $x \in [0, \infty]$ to $t \in [0,1]$ then $t \in [0,1]$ to $v \in [-1,1]$ by monomial transformation Eq. (2) to find approximate solution of Eq. (1)has been made by mixed quadrature rule with extracting end point singularity automatically. The construction of mixed quadrature rule of degree of precision eleven is elaborated in the following sections.

Product integral performs an important role in theory of survival analysis , Markov process and quantum Mechanics. Product integrals are similar to classical Rieman integral of a function $f:[a,b] \to R$ by the relation

$$\int_a^b f(x) dx = \lim_{\Delta x \to 0} \sum f(x_i) \Delta x_i$$

where the limit is taken over all the partitions of interval $[a,b]$,whose norm approaches zero. But in product integral it has been taken the limit of product instead of the limit of a sum the commonly product integrals are the following two types

Type-1 $$\prod_a^b f(x)^{dx} = \lim_{\Delta x \to 0} \prod f(x_i)^{\Delta x} = \exp\left(\int_a^b \ln(f(x)) dx \right)$$

which is called Geometric integral

Type-2 $\prod_a^b (1+f(x)dx) = \lim_{\Delta x \to 0} \prod (1+f(x_i)dx) = \exp\left(\int_a^b f(x)dx\right)$

Basic properties of Product Integral

a) $\prod_a^b c^{dx} = c^{b-a}$, b) $\prod_a^b f(x)^{kdx} = \left(\prod_a^b f(x)^{dx}\right)^k$

c) $\prod_a^b (c^{f(x)})dx = c^{\int_a^b f(x)dx}$, d) $\prod_a^b x^{dx} = \left(\frac{s}{e}\right)^s$

The fundamental theorem

$$\prod_a^b f'^*(x)^{dx} = \prod_a^b \exp\left(\frac{f'(x)}{f(x)}dx\right) = \frac{f(b)}{f(a)}$$

Where $f'^*(x)$ is geometric derivative.

$$f^*(x) = \lim_{h\to 0}\left[\frac{f(x+h)}{f(x)}\right]^{\frac{1}{h}} = e^{\frac{f'(x)}{f(x)}}$$

For continuous function $f(x)$, $\int_a^b f(x)dx = \prod_a^b f(x)^{dx}$

$$\ln \prod_a^b p(x)^{dx} = \int_a^b \ln p(x)dx$$

In Section 2, the mixed quadrature rule of degree of precision eleven $R_{ST4GL2GL3GL4GL5}(f)$ has been constructed by Keeping in the mind,(D.Das and R.B.Dash[4]), (S.R.Jena and S. Mishra[7]) and (R.N. Das and G. Pradhan [8]) (Gradimir V.Milovanoic,et.al. [5], [1]) by taking convex combination $R_{ST4GL2GL3GL4GL5}(f)$ and $R_{GL5}(f)$ each rule of precision nine. Section 3 contains an error analysis of the proposed mixed quadrature rule . In section 4, the rule has been numerically verified taking suitable examples. The conclusion has been drawn in Section-5 and our result of Table-1and Table-2 is compared with result of [10] in Table-3,Table-4 respectively.

2. Consruction of Mixed Quadrature Rules

Keeping in mind Jena S.R and Nayak D.[6], (Sameet Deshpande, Subham Kumar et.al[9]) ,(A. Pati and Rajani B. Dash et. al.[2]) our proposed mixed quadrature rule $I_{ST4GL2GL3GL4GL5}(f)$ of degree of precision eleven is obtained by combining $I_{ST4GL2GL3GL4}(f)$ with Gauss Legendre-5 point rule $I_{GL5}(f)$ which is given in the theorem 2.1

THEOREM-2.1

Let f(x) be sufficiently differentiable function in $[-1,1]$. If the error $E_{ST4GL2GL3GL4}(f)$ and $E_{GL5}(f)$ due to the mixed quadrature rule of $I_{ST4GL2GL3GL4}(f)$ and the Gauss Legendre- 5 point rule $I_{GL5}(f)$ respectively ,then they together form a proposed mixed quadrature rule of degree of precision eleven $I_{ST4GL2GL3GL4GL5}(f)$.

$$I(f) = \int_{-1}^{1} f(x)dx \cong \frac{1}{112471}[10880 I_{ST4GL2GL3GL4}(f) + 101591 Q_{GL5}(f)] +$$
$$\frac{1}{112471}[10880 E_{ST4GL2GL3GL4}(f) + 101591 E_{GL5}(f)]$$

Where

$$I_{ST4GL2GL3GL4GL5}(f)=\frac{1}{112471}\left[10880I_{ST4GL2GL3GL4}(f)+101591I_{GL5}(f)\right]$$

and

$$E_{ST4GL2GL3GL4GL5}(f)=\frac{1}{112471}\left[10880E_{ST4GL2GL3GL4}(f)+101591E_{GL5}(f)\right]$$

Proof:

$$I(f)=\int_{-1}^{1} f(x)dx \cong \frac{1}{51}\left[301I_{GL4}(f)-250I_{ST4GL2GL3}(f)\right]+ \frac{1}{51}\left[301I_{GL4}(f)-250I_{ST4GL2GL3}(f)\right] \tag{3}$$

where

$$I_{ST4}(f)=\frac{1}{12}\left[11\left\{f\left(-\frac{3}{5}\right)+f\left(\frac{3}{5}\right)\right\}+\left\{f\left(-\frac{1}{5}\right)+f\left(\frac{1}{5}\right)\right\}\right]$$

$$I_{GL2}(f)=f\left(-\sqrt{\frac{3}{5}}\right)+f\left(\sqrt{\frac{3}{5}}\right)$$

$$I_{GL3}(f)=\frac{1}{9}\left[5\left\{f\left(-\sqrt{\frac{3}{5}}\right)+f\left(\sqrt{\frac{3}{5}}\right)\right\}+8f(0)\right]$$

$$I_{GL4}(f)=\frac{1}{36}\left[\left(18+\sqrt{30}\right)\{f(-\alpha)+f(\alpha)\}+\left(18-\sqrt{30}\right)\{f(-\beta)+f(\beta)\}\right]$$

where $\alpha=\sqrt{\dfrac{3-2\sqrt{\frac{6}{5}}}{7}}$ and $\beta=\sqrt{\dfrac{3+2\sqrt{\frac{6}{5}}}{7}}$

and the Gauss-Legendre-5 point rule $I_{GL5}(f)$ is

$$I(f)=\int_{-1}^{1} f(x)\,dx \cong R_{GL5}(f)=\frac{1}{900}\left[\begin{array}{l}\left(322+13\sqrt{70}\right)\{f(-\alpha)+f(\alpha)\}+\\ \left(322-13\sqrt{70}\right)\{f(-\beta)+f(\beta)\}+512f(0)\end{array}\right] \tag{4}$$

where

$$\alpha=\sqrt{\frac{5-2\sqrt{\frac{10}{7}}}{9}} \text{ and } \beta=\sqrt{\frac{5+2\sqrt{\frac{10}{7}}}{9}}$$

Each of the rule $I_{ST4GL2GL3GL4}(f)$ and $I_{GL5}(f)$ is of precision nine. Let $E_{ST4GL2Gl3GL4}(f)$ and $E_{GL5}(f)$ denotes the errors in approximating the integral $I(f)$ by the rules $I_{ST4GL2GL3GL4}(f)$ and $I_{GL5}(f)$ respectively.

$$I(f)=I_{ST4GL2GL3GL4}(f)+E_{ST4GL2GL3GL4}(f) \tag{5}$$

$$I(f)=I_{GL5}(f)+E_{GL5}(f) \tag{6}$$

Using Maclaurin's expansion of functions in Eq. (5) and (6), we get

$$E_{ST4GL2GL3GL4}(f)\cong-\frac{14513}{530145\times10!}f^{x}(0)$$

$$E_{GL5}(f)\cong\frac{128}{11\times9^{2}\times7^{2}\times10!}f^{x}(0)+\frac{2706048}{13\times9^{6}\times7^{2}\times12!}f^{xii}(0)+\text{........}$$

Multiplying Eq. (5) and (6) by $\left(\frac{128}{7}\right)$ and $\left(\frac{14513}{85}\right)$ respectively and adding the resulting equations, we have

$$I(f)=\int_{-1}^{1} f(x)dx \cong \frac{1}{112471}\left[10880 I_{ST4GL2GL3GL4}(f)+101591 I_{GL5}(f)\right]+$$

$$\frac{1}{112471}\left[10880 E_{ST4GL2GL3GL4}(f)+101591 E_{GL5}(f)\right]$$

Then

$$I(f)=I_{ST4GL2GL3GL4GL5}(f)+E_{ST4GL2GL3GL4GL5}(f) \tag{7}$$

Where

$$I_{ST4GL2GL3GL4GL5}(f)=\frac{1}{112471}\left[10880 I_{ST4GL2GL3GL4}(f)+101591 I_{GL5}(f)\right]$$

This is the desired mixed quadrature rule of precision eleven for the approximate evaluation of I(f). The truncation error generated in this approximation is given by

$$E_{ST4GL2GL3GL4GL5}(f)=\frac{1}{112471}\left[10880 E_{ST4GL2GL3GL4}(f)+101591 E_{GL5}(f)\right] \tag{8}$$

Corollary-2.1

If $E_{ST4GL2GL3GL4GL5}(f)=\frac{1}{112471}\left[10880 E_{ST4GL2GL3GL4}(f)+101591 E_{GL5}(f)\right]$

Then the truncated error bound is given by

$$\left|E_{ST4GL2GL3GL4GL5}(f)\right| \cong \frac{274910122368}{38074573352907\times 12!}\left|f^{xii}(0)\right|$$

Proof:

Using Maclaurin's theorem in Eq. (8) , we have

$$\left|E_{ST4GL2GL3GL4GL5}(f)\right| \cong \frac{274910122368}{38074573352907\times 12!}\left|f^{xii}(0)\right| \tag{9}$$

3. Error Analysis

Theorem 3.1

The error bound for the truncation error

$E_{ST4GL2GL3GL4GL5}(f)=I(f)-I_{ST4GL2GL3GL4GL5}(f)$ is given by

$$\left|E_{ST4GL2GL3GL4GL5}(f)\right| \le \frac{119734M}{22606671\times 10!}$$

where $M=\max_{-1\le x\le 1}\left|f^{xi}(x)\right|$

Proof: we have

$$E_{ST4GL2GL3GL4}(f)\cong \frac{59867}{22606671\times 10!} f^{x}(\eta_1), \quad where\ \eta_1\in[-1,1]$$

$$E_{GL5}(f)=\frac{59867}{22606671\times 10!} f^{x}(\eta_2), \quad where\ \eta_2\in[-1,1]$$

where $K=\max_{-1\le x\le 1}\left|f^{x}(x)\right|$ and $k=\min_{-1\le x\le 1}\left|f^{x}(x)\right|$. As $f^{x}(x)$ is continuous & $[-1,1]$ is compact .Hence there exists points b and a in the interval $[-1,1]$ such that $K=f^{x}(b)$ and $k=f^{x}(a)$.Thus by Conte & Boor[3]

$$\left|E_{ST4GL2GL3GL4GL5}(f)\right| \cong \frac{59867}{22606671\times 10!}\left\{f^{x}(b)-f^{x}(a)\right\}$$

$$= \frac{59867}{22606671\times 10!}\int_{-1}^{1} f^{xi}(x)dx$$

$$= \frac{59867}{22606671\times 10!}(b-a)\ \ f^{xi}(\xi)\ \ for\ \ some\ \ \xi\in[-1,1]$$

By mean value theorem $|b-a|\le 2$

Then $\left|E_{ST4GL2GL3GL4GL5}(f)\right| \le \frac{119734}{22606671\times 10!} f^{xi}(\xi)$

$$\left|E_{ST4GL2GL3GL4GL5}(f)\right| \le \frac{119734M}{22606671\times 10!} \qquad \text{where } M=\max_{-1\le x\le 1}\left|f^{xi}(x)\right|$$

4. Numerical Verification

T his section contains numerical experiments on following two integrals.

$$I_1 == \int_0^{\infty} 10e^{-2x}\left(x^2+1\right)\ dx = \int_0^1 10y\left[(\log y)^2+1\right]\frac{dy}{2} = 7.5$$

$$I_2 = \int_0^{\infty} x^{\frac{-3}{2}} \sin\left(\frac{1}{x}\right)\ dx = \int_0^1 \frac{\sin t}{\sqrt{t}}\, dt = 0.62053661$$

TABLE-1(Approximate solution of Example-1)

Quadrature rule	$I_1(Approx)$	$\lvert Error\rvert$
$I_{ST4GL2GL3GL4}(f)$	7.486392784239775	0.013607215760225
$I_{GL5}(f)$	7.520269379676130	0.020269379676130
$I_{ST4GL2GL3GL4GL5}(f)$	7.516992291730371	0.016992291730371

TABLE-2 (Approximate solution of Example-1)

Quadrature rule	$I_2(Approx)$	$\lvert Error\rvert$
$I_{ST4GL2GL3GL4}(f)$	7.486392784239775	0.000251295487213
$I_{GL5}(f)$	0.620285384512787	0.000251225487213
$I_{ST4GL2GL3GL4GL5}(f)$	0.621081279799653	0.000544669799653

TABLE-3(Comparison of the mixed method with Romberg method for problem I_1)

N	C	Exact Value	Romberg Method	Mixed Method	Error 1 (Exact and Romberg)	Error 2 (Exact and Mixed Method)
16	1	7.50	7.429370880	7.467471252	0.070629120	0.032528748
	2	7.50	7.473458648	7.466256983	0.026541352	0.013743017
32	1	7.50	7.476584315	7.489618657	0.023415685	0.010381343
	2	7.50	7.492322326	7.490176929	0.007677674	0.003823071
64	1	7.50	7.492510676	7.496606434	0.007489324	0.003393566
	2	7.50	7.497819662	7.497197233	0.002180338	0.001802767
256	1	7.50	7.499290705	7.499647302	0.000709295	0.000352698

	2	7.50	7.499831319	7.499781475	0.000168681	0.000118525

N= Numbers of points, C=Number of column in Romberg's method

TABLE-4(Comparison of the mixed method with Romberg method for problem I_2)

N	C	Exact Value	Romberg Method	Mixed Method	Error 1 (Exact and Romberg)	Error 2 (Exact and Mixed Method)
16	1	0.62053661	0.61732721	0.61878139	0.00320938	0.00175515
	2	0.62053661	0.61926835	0.61899556	0.00126824	0.00114104
32	1	0.62053661	0.61939787	0.61989174	0.00113872	0.00064486
	2	0.62053661	0.62008810	0.61999152	0.00044849	0.00034508
64	1	0.62053661	0.62013298	0.62030369	0.00040361	0.00023291
	2	0.62053661	0.62037801	0.62034484	0.00015858	0.00011276
256	1	0.62053661	0.62048602	0.62050695	0.00005057	0.00002965
	2	0.62053661	0.62051683	0.62051250	0.00001976	0.00001410

N= Numbers of points, C=Number of column in Romberg's method

5. Conclusion

In this paper we have constructed mixed quadrature for the product integral in the infinite integral. From the numerical verification it is observed that our mixed quadrature rule given in Table-1 and Table-2 gives better approximation than result obtained in [10] taking only 17-point functional evaluation. Our method gives same result taking minimum number of functional evaluation whereas it requires 32 points and 64 points functional evaluation in Romberg and Mixed method in Table-3 and Table-4. Generally the computational result has showed that the proposed mixed quadrature rule is performed slightly better when compared to that standard Romberg method and mixed method with respect to functional point evaluation.

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