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Application of mixed quadrature in adaptive environment over a square domain

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Abstract: This paper contains a quadrature rule for double integrals of precision-7 has been constructed taking two rules of precision-5.In adaptive environment the rule has been numerically experimented on different integrals as well as it has been implemented to evaluate the line integral for any vector function through the Green's theorem. It has been found a good agreement to that of Clenshaw-Curtis five point rule. An error analysis has also been made.

Keywords: Degree of precision, Maclaurin's series, Error bound, Adaptive quadrature scheme, Green's theorem.
MSC: 2010 : 65D30, 65D32

1. Introduction

Since decade years so many researchers [2,7,10,11] have been working on the integral

$$I(f)=\int_a^b f(s)ds \qquad (1.1)$$

Which is real in nature for calculation of single variables . Here in this article we have used a quadrature rule on the new scheme i.e. adaptive method for the evaluation of integral (1.1) for double variables.
The main principle behind adaptive quadrature is the additive property of a definite integral. If $h \in [m,n]$ then

$$A+B=C$$

Where $A=\int_m^p f(s)ds$, $B=\int_p^n f(s)ds$, $C=\int_m^n f(s)ds$

The idea is to compute a real integrable function f , an interval $[m,n]$ for a prescribed tolerance ε , the integral $\int_m^n f(s)ds = I$ so that $|C-I| \le \varepsilon$. In adaptive integration, the points at which the integrand is evaluated are chosen in accordance with the nature of the integrand. The fundamental principle is to get the sum which gives the appropriate result with approximate evaluation of two integrals for a specified tolerance. If not, we can recursively apply the additive property to each of the intervals $[m,p]$ and $[p,n]$. Adaptive subdivision of course has the geometrical meaning. It seems intuitive that points should be concentrated in regions where the integrand is badly behaved. The whole interval rules can take no direct account of this.

Keeping the above facts in mind Clenshaw-Curtis quadrature rule and a quadrature rule of higher precision (mixed quadrature rule) has been constructed with the help of two lower precision rules which is given in sec-2 and sec-3 respectively. Sec-4 contains the error analysis. The mixed quadrature rule is tested numerically on different definite integrals which occur in sec-5. The application and the conclusion follow in sec-6 and sec-7 respectively.

2. Clenshaw-Curtis quadrature rule

Here $f(v)$ can be approximated by Clenshaw-Curtis method [1] for evaluation of $I(f)$.

$$I(f)=\int_a^b f(v)dv = h\int_{-1}^{1} f(\alpha+hs)ds$$

$$I \approx I_n = h\sum_{r=0}^{n} a_r \int_{-1}^{1} T_r(s)ds \qquad (2.1)$$

Where $T_r(s) = \cos(r\cos^{-1}(s)), r \geq 0$ the Chebyshev polynomial of degree n

And $T_r(s_i) = \cos(r\cos^{-1}(s_i))$, $s_i = \cos\left(\frac{i\pi}{n}\right), i = 0,1,........, n$

$$T_r(s_i) = \cos\left(\frac{ri\pi}{n}\right)$$

$$a_r = \begin{cases} \left(\dfrac{2\times\sum_{i=0}^{n} f(\alpha + hs_i)\ T_r(s_i)}{n}\right) & r = 0,1,..., n-1 \\ \left(\dfrac{\sum_{i=0}^{n} f(\alpha + hs_i)\, T_r(s_i)}{n}\right) & r = n \end{cases}$$

Substituting a_r and $T_r(s)$ in eqn (2.1)

$$I_n = h\sum_{r=0}^{n} \frac{2}{n}\sum_{i=0}^{n} f(\alpha + hs_i)T_r(s_i)\int_{-1}^{1} T_r(s)dx$$

Since $\int_{-1}^{1} T_r(s)ds = -\dfrac{2}{r^2 - 1}$ $(r = even)$

$$I_n = h\sum_{i=0}^{n}\left(-\frac{4}{n}\sum_{r=0}^{n}\frac{1}{r^2 - 1}T_r(s_i)\right) f(\alpha + hs_i) \quad , r = even$$

Specially for $n = 4,$

$$I_4 = \frac{h}{15}[D + 8E + 12G] \qquad (2.2)$$

Where $D = f(\alpha - h) + f(\alpha + h), E = \left\{ f\left(\alpha - \frac{h}{\sqrt{2}}\right) + f\left(\alpha + \frac{h}{\sqrt{2}}\right)\right\}, G = f(\alpha)$

3. Construction of the quadrature rule.

Here the two rules i.e $R_{CC5}(f)$(Clenshaw-Curtis 5-point) rule and $R_{GL3}(f)$(Gauss Legendre 3-point) rule each of precision-5 have been combined to form the rule $R_{CC5GL3}(f)$(mixed quadrature) rule of precision-7. A transformation $-1 \leq s \leq 1$, $-1 \leq u \leq 1$ has occurred taking the substitution $2s = (1+t)b + (1-t)a$, $2u = (1+t)b + (1-t)a$.

From eqn (2.2)

$$I(f) = \int_{-1}^{1}\int_{-1}^{1} f(s,u)ds\,du \cong R_{CC5}(f)$$

$$= \frac{1}{225}\left[\begin{array}{l} \left\{ f(1,1)+ f(1,-1)+8f\left(1,\frac{1}{\sqrt{2}}\right)+8f\left(1,-\frac{1}{\sqrt{2}}\right)+12f(1,0)\right\}+ \\ \left\{ f(-1,1)+ f(-1,-1)+8f\left(-1,\frac{1}{\sqrt{2}}\right)+8f\left(-1,-\frac{1}{\sqrt{2}}\right)+12f(-1,0)\right\}+ \\ 8\left\{ f\left(\frac{1}{\sqrt{2}},1\right)+ f\left(\frac{1}{\sqrt{2}},-1\right)+8f\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)+8f\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)+12f\left(\frac{1}{\sqrt{2}},0\right)\right\}+ \\ 8\left\{ f\left(-\frac{1}{\sqrt{2}},1\right)+ f\left(-\frac{1}{\sqrt{2}},-1\right)+8f\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)+8f\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)+12f\left(-\frac{1}{\sqrt{2}},0\right)\right\}+ \\ 12\left\{ f(0,1)+ f(0,-1)+8f\left(0,\frac{1}{\sqrt{2}}\right)+8f\left(0,-\frac{1}{\sqrt{2}}\right)+12f(0,0)\right\} \end{array}\right] \quad (3.1)$$

$$I(f)=\int_{-1}^{1}\int_{-1}^{1} f(s,u)\,ds\,du \cong R_{GL3}(f)=\frac{1}{81}\left[\begin{array}{l} 5\left\{5f\left(-\sqrt{\frac{3}{5}},-\sqrt{\frac{3}{5}}\right)+8f\left(-\sqrt{\frac{3}{5}},0\right)+5f\left(-\sqrt{\frac{3}{5}},\sqrt{\frac{3}{5}}\right)\right\}+ \\ 8\left\{5f\left(0,-\sqrt{\frac{3}{5}}\right)+8f(0,0)+5f\left(0,\sqrt{\frac{3}{5}}\right)\right\}+ \\ 5\left\{5f\left(\sqrt{\frac{3}{5}},-\sqrt{\frac{3}{5}}\right)+8f\left(\sqrt{\frac{3}{5}},0\right)+5f\left(\sqrt{\frac{3}{5}},\sqrt{\frac{3}{5}}\right)\right\} \end{array}\right] \quad (3.2)$$

Using Maclaurin's expansion of $I(f)=\int_{-1}^{1}\int_{-1}^{1} f(s,u)\,ds\,du$,

$$\begin{aligned} I(f) &= 4f_{0,0}(0,0)+\frac{2}{3}\left[f_{2,0}(0,0)+f_{0,2}(0,0)\right]+\frac{1}{30}\left[f_{4,0}(0,0)+f_{0,4}(0,0)\right] \\ &+\frac{1}{9}f_{2,2}(0,0)+\frac{1}{180}\left[f_{4,2}(0,0)+f_{2,4}(0,0)\right]+\frac{4}{7!}\left[f_{6,0}(0,0)+f_{0,6}(0,0)\right]+ \\ &+\frac{1}{3600}f_{4,4}(0,0)+\frac{1}{7560}\left[f_{6,2}(0,0)+f_{2,6}(0,0)\right]+\frac{4}{9!}\left[f_{8,0}(0,0)+f_{0,8}(0,0)\right]+\ldots \end{aligned} \quad (3.3)$$

Theorem-3.1

In this theorem a mixed quadrature rule and its error have been determined.

$$I(f)=R_{CC5GL3}(f)+E_{CC5GL3}(f)$$

$$R_{CC5GL3}(f)=\frac{1}{7}\left[12R_{CC5}(f)-5R_{GL3}(f)\right]$$

(3.4)

$$E_{CC5GL3}(f)=\frac{1}{7}\left[12E_{CC5}(f)-5E_{GL3}(f)\right] \quad (3.5)$$

Where $E_{CC5}(f)$ and $E_{GL3}(f)$ are the errors due to $R_{CC5}(f)$ and $R_{GL3}(f)$ respectively.

Proof: Expanding each term of eqn (3.1) and eqn (3.2) using Maclaurin's series

$$I(f)=R_{CC5}(f)+E_{CC5}(f) \quad (3.6)$$

and $I(f)=R_{GL3}(f)+E_{GL3}(f)$ (3.7)

Where

$$R_{CC5}(f)=4f_{0,0}(0,0)+\frac{2}{3}\left[f_{2,0}(0,0)+f_{0,2}(0,0)\right]+\frac{1}{30}\left[f_{4,0}(0,0)+f_{0,4}(0,0)\right]$$
$$+\frac{1}{9}f_{2,2}(0,0)+\frac{1}{180}\left[f_{4,2}(0,0)+f_{2,4}(0,0)\right]+\frac{8}{15\times 6!}\left[f_{6,0}(0,0)+f_{0,6}(0,0)\right]+ \tag{3.8}$$
$$+\frac{1}{3600}f_{4,4}(0,0)+\frac{224}{45\times 8!}\left[f_{6,2}(0,0)+f_{2,6}(0,0)\right]+\frac{2}{5\times 8!}\left[f_{8,0}(0,0)+f_{0,8}(0,0)\right]+\ldots$$

$$R_{GL3}(f)=4f_{0,0}(0,0)+\frac{2}{3}\left[f_{2,0}(0,0)+f_{0,2}(0,0)\right]+\frac{1}{30}\left[f_{4,0}(0,0)+f_{0,4}(0,0)\right]$$
$$+\frac{1}{9}f_{2,2}(0,0)+\frac{1}{180}\left[f_{4,2}(0,0)+f_{2,4}(0,0)\right]+\frac{12}{25\times 6!}\left[f_{6,0}(0,0)+f_{0,6}(0,0)\right]+ \tag{3.9}$$
$$+\frac{1}{3600}f_{4,4}(0,0)+\frac{112}{25\times 8!}\left[f_{6,2}(0,0)+f_{2,6}(0,0)\right]+\frac{2916}{78125\times 8!}\left[f_{8,0}(0,0)+f_{0,8}(0,0)\right]+\ldots$$

$$E_{CC5}(f)=I(f)-R_{CC5}(f)$$
$$=\frac{4}{105\times 6!}\left[f_{6,0}(0,0)+f_{0,6}(0,0)\right]+\frac{16}{45\times 8!}\left[f_{6,2}(0,0)+f_{2,6}(0,0)\right] \tag{3.10}$$
$$+\frac{2}{45\times 8!}\left[f_{8,0}(0,0)+f_{0,8}(0,0)\right]$$

$$E_{GL3}(f)=I(f)-R_{GL3}(f)$$
$$=\frac{16}{175\times 6!}\left[f_{6,0}(0,0)+f_{0,6}(0,0)\right]+\frac{64}{75\times 8!}\left[f_{6,2}(0,0)+f_{2,6}(0,0)\right] \tag{3.11}$$
$$+\frac{286256}{703125\times 8!}\left[f_{8,0}(0,0)+f_{0,8}(0,0)\right]$$

Multiplying $\left(\frac{4}{5}\right)$ in eqn (3.3) and $\left(\frac{1}{3}\right)$ in eqn (3.4) and subtracting eqn (3.4) from eqn (3.3)

$$I(f)=R_{CC5GL3}(f)+E_{CC5GL3}(f) \tag{3.12}$$

Where $$R_{CC5GL3}(f)=\frac{1}{7}\left[12R_{CC5}(f)-5R_{GL3}(f)\right] \tag{3.13}$$

$$E_{CC5GL3}(f)=\frac{1}{7}\left[12E_{CC5}(f)-5E_{GL3}(f)\right] \tag{3.14}$$

4. Error analysis

Theorem-4.1

The error bound is $\left|E_{CC5GL3}(f)\right|\cong\left|\frac{211256}{140625\times 8!}\left[f_{8,0}(0,0)+f_{0,8}(0,0)\right]\right|$.

Proof- From eqn (3.12),

$$I(f)=R_{CC5GL3}(f)+E_{CC5GL3}(f)$$
$$R_{CC5GL3}(f)=\frac{1}{7}\left[12R_{CC5}(f)-5R_{GL3}(f)\right]$$

$$E_{CC5GL3}(f)=\frac{1}{7}\left[12E_{CC5}(f)-5E_{GL3}(f)\right]$$

Hence $\left|E_{CC5GL3}(f)\right| \cong \left|\frac{211256}{140625\times 8!}\left[f_{8,0}(0,0)+f_{0,8}(0,0)\right]\right|$

Theorem-4.2

$$\left|E_{CC5GL3}(f)\right| \le \frac{32M}{245\times 6!}$$

Where $M = \max\limits_{\substack{-1\le x\le 1\\ -1\le y\le 1}}\left|f_{7,0}(s,*)+f_{0,7}(*,u)\right|$

Proof: We have $E_{CC5}(f) = \frac{4}{105\times 6!}\left[f_{6,0}(\eta_2,0)+f_{0,6}(0,\eta_2)\right]$

$$E_{GL3}(f) = \frac{16}{175\times 6!}\left[f_{6,0}(\eta_1,0)+f_{0,6}(0,\eta_1)\right] \quad \text{where} \quad \eta_1,\eta_2 \in [-1,1]$$

$$\begin{aligned}
E_{CC5GL3}(f) &= \frac{1}{7}\left[12E_{CC5}(f)-5E_{GL3}(f)\right]\\
&= \frac{16}{245\times 6!}\left[f_{6,0}(\eta_2,0)+f_{0,6}(0,\eta_2)-f_{6,0}(\eta_1,0)-f_{0,6}(0,\eta_1)\right]\\
&= \frac{16}{245\times 6!}\left[\int_{\eta_1}^{\eta_2} f_{7,0}(s,0)\,ds+\int_{\eta_1}^{\eta_2} f_{0,7}(0,u)\,du\right]\\
&= \frac{16}{245\times 6!}\int_{\eta_1}^{\eta_2}\int_{\eta_1}^{\eta_2}\left[f_{7,0}(s,*)+f_{0,7}(*,u)\right]ds\,du
\end{aligned}$$

$$\begin{aligned}
\left|E_{CC5GL3}(f)\right| &\le \frac{16M}{245\times 6!}\left|\eta_2-\eta_1\right|\\
&\le \frac{32M}{245\times 6!} \qquad \textit{for } \left|\eta_2-\eta_1\right|\le 2 \ (\text{C.Conte and D.Boor}[8])
\end{aligned}$$

Where $M = \max\limits_{\substack{-1\le x\le 1\\ -1\le y\le 1}}\left|f_{7,0}(s,*)+f_{0,7}(*,u)\right|$

5. Numerical verification

The integrals under considerations are

$$I_1 = \int_{-1}^{1}\int_{-1}^{1} e^{s+u}\,du\,ds,\ I_2 = \int_0^1\int_0^1 \sin^2(s+u)\,du\,ds,\ I_3 = \int_0^1\int_0^1 \cos^2(s+u)\,du\,ds,\ I_4 = \int_0^1\int_0^1 \frac{1}{4+s+u}\,du\,ds,$$

$$I_5 = \int_0^2\int_0^2 sue^{(s^2-u^2)}\,du\,ds.$$

Table-1(comparison between exact value and approximate value with stopping criterion ε)

Exact value of the integrals	$R_{CC5}(f)$ *by adaptive method*	*No of Intervals for* $R_{CC5}(f)$	$R_{CC5GL3}(f)$ *by adaptive method*	*No of Intervals for* $R_{CC5GL3}(f)$	*absolute error* (ε)
I_1=5.52439138 2167263	5.52439134680 9600	5	5.524391391356736	1	$\varepsilon_1 = 0.00000003$
I_2=0.64733125 6528834	0.64733125748 9143	5	0.647331256791722	1	$\varepsilon_2 = 0.0000000009$

I_3=0.352668743471166	0.352668742510857	5	0.352668743208278	1	$\varepsilon_3 = 0.0000000009$
I_4=0.201355135506889	0.201355135506187	8	0.201355135507072	1	$\varepsilon_4 = 0.0000000000007$
I_5=13.154116418008243	13.1540204272058572	8	13.154144863156123	3	$\varepsilon_5 = 0.00009$

6. Application

We have made a transformation of double integral into a single integral by Green's theorem over a square domain D in adaptive scheme.

Evaluation of line integral by Green's theorem in adaptive environment.
Green's theorem

Let $F = [F_1, F_2] = F_1 i + F_2 j$ .Then

$$\iint_R \left(\frac{\partial F_2}{\partial s} - \frac{\partial F_1}{\partial u}\right) ds du = \oint_C F.dr = \oint_C (F_1 ds + F_2 du) [9]$$

$$\Rightarrow \oint_C F.dr = \iint_R \left(\frac{\partial F_2}{\partial s} - \frac{\partial F_1}{\partial u}\right) ds du \cong R_{CC5GL3}(f) = \frac{12}{7} R_{CC5}(f) - \frac{5}{7} R_{GL3}(f)$$

$$= \frac{4}{525}\left[\begin{array}{l} \left\{ f(1,1) + f(1,-1) + 8f\left(1,\frac{1}{\sqrt{2}}\right) + 8f\left(1,-\frac{1}{\sqrt{2}}\right) + 12f(1,0)\right\} + \\ \left\{ f(-1,1) + f(-1,-1) + 8f\left(-1,\frac{1}{\sqrt{2}}\right) + 8f\left(-1,-\frac{1}{\sqrt{2}}\right) + 12f(-1,0)\right\} + \\ 8\left\{ f\left(\frac{1}{\sqrt{2}},1\right) + f\left(\frac{1}{\sqrt{2}},-1\right) + 8f\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right) + 8f\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right) + 12f\left(\frac{1}{\sqrt{2}},0\right)\right\} + \\ 8\left\{ f\left(-\frac{1}{\sqrt{2}},1\right) + f\left(-\frac{1}{\sqrt{2}},-1\right) + 8f\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right) + 8f\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right) + 12f\left(-\frac{1}{\sqrt{2}},0\right)\right\} + \\ 12\left\{ f(0,1) + f(0,-1) + 8f\left(0,\frac{1}{\sqrt{2}}\right) + 8f\left(0,-\frac{1}{\sqrt{2}}\right) + 12f(0,0)\right\} \end{array}\right]$$

$$- \frac{5}{567}\left[\begin{array}{l} 5\left\{5f\left(-\sqrt{\frac{3}{5}},-\sqrt{\frac{3}{5}}\right) + 8f\left(-\sqrt{\frac{3}{5}},0\right) + 5f\left(-\sqrt{\frac{3}{5}},\sqrt{\frac{3}{5}}\right)\right\} + \\ 8\left\{5f\left(0,-\sqrt{\frac{3}{5}}\right) + 8f(0,0) + 5f\left(0,\sqrt{\frac{3}{5}}\right)\right\} + \\ 5\left\{5f\left(\sqrt{\frac{3}{5}},-\sqrt{\frac{3}{5}}\right) + 8f\left(\sqrt{\frac{3}{5}},0\right) + 5f\left(\sqrt{\frac{3}{5}},\sqrt{\frac{3}{5}}\right)\right\} \end{array}\right] \quad (6.1)$$

Here we integrate along the entire boundary C of D in adaptive environment.

Numerical test

Here we compare $R_{CC5GL3}(f)$ rule with $R_{CC5}(f)$ rule for approximate evaluation of line integral through Green's theorem for several vector function F in adaptive method. The line integrals are

(i). $F = [e^s \cos u, su^3]$, $D: 1 \le s, u \le 3$

$$I_6 = \oint_C F.dr = \iint_D \left(\frac{\partial}{\partial s}(su^3) - \frac{\partial}{\partial u}(e^s \cos u)\right) ds du = \int_1^3\int_1^3 (u^3 + e^s \sin u) du\, ds$$

(ii). $F = [-\cosh u, \sinh s]$, $D: 1 \le s, u \le 2$

$$I_7 = \oint_C F.dr = \iint_D \left(\frac{\partial}{\partial s}(\sinh s) - \frac{\partial}{\partial u}(-\cosh u) \right) ds du = \int_1^2 \int_1^2 (\cosh s + \sinh u) du\, ds$$

Table-2 (comparison of exact value of line integral with approximate value with stopping criterion ε)

Exact value	$R_{CC5}(f)$ *by adaptive method*	*No of Intervals for* $R_{CC5}(f)$	$R_{CC5GL3}(f)$ *by adaptive method*	*No of Intervals for* $R_{CC5GL3}(f)$	*absolute error* (ε)
I_6=66.577020204609269	66.577020206173898	5	66.577020250430436	1	$\varepsilon_6 = 0.00000004$
I_7=4.670774270471605	4.670774270236460	5	4.670774270487181	1	$\varepsilon_7 = 0.0000000002$

7. Conclusion

The effectiveness of $R_{CC5GL3}(f)$ rule has been observed from seven numerical texts provided in Table-1 and Table-2 in adaptive scheme. The rule $R_{CC5GL3}(f)$ takes the less number of iterations for approximate evaluation of any integral in adaptive scheme to that of $R_{CC5}(f)$ rule. This work may be extended for any domain D instead of a square domain.

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