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Mid –Truncated Two-Parameter Lindley Distribution and Its Applications in Order Statistics

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Abstract: In this paper we define the Mid-truncated two-parameter Lindley distributionand derive some of its statistical properties such as moments, moment generating function and characteristicfunction. We have also obtained the recurrence relations for single and product moments of order statistics in a random sample of size n drawn fromMid-Truncated two-parameter Lindley distribution.

Keywords: Order statistics; Mid-truncated two-parameter Lindley distribution; Single moments; Product moments; Recurrence relations; Incomplete Gamma function.

1. Introduction

The truncated distributions are quite effectively used where a random variable is restricted to be observed on some range and these situations are common in various fields. For instance, in survival analysis, failures during the warranty period may not be counted. Items may also be replaced after certain time following the replacement policy, so that failures of the item are ignored. Therefore, many researchers were being attracted to the problem of analyzing such truncated data encountered in various disciplines.Many authors like Malik (1967), Balakrishnan and Joshi (1981), Balakrishnan et al. (1988),Saran and Pushkarna(1999 a, b), etc. have obtained several results for the single and product moments of order statistics from the un-truncated, left truncated, right truncated and doubly truncated distributions. Mohie El-Din and Sultan (1993) have obtained recurrence relations for moments of order statistics from doubly truncated continuous distributions.But there exist life models which do not obey the complete or truncated distribution, for example, in Microbiology, when a bacterial strain is inoculated into a liquid growth medium, the population is counted at intervals, it is possible to plot a bacterial growth curve. There are three basic phases of growth: the log, stationary, and death phases. During log phase, the bacterial cells are most metabolically active and are preferred for industrial purposes. During stationary phase we find that the number of cells will still remain constant due to the number of bacterial death balances with the number of new cells. Consequently during this phase no investigation is required. After the stationary phase the most bacterial cells will diebecause of the exclusion of nutrientsand would lead to accumulation of waste products. In such live models mid-truncated distributions are quite effectively used where the random variable is restricted to be observed on some sub intervals of the given specified range.(cf. Okasha et al. (2011) and Mohie El-Din et al. (2013)).

Monitoring the wide applicability of the truncated distributions, we propose the mid-truncation in the two-parameter Lindley distribution. Lindley distribution was introduced by D.V. Lindley (1958) in the context of Bayesian analysis as a counter example of fiducial statistics and is a mixture of exponential and gamma distributions. A detailed study about its important mathematical and statistical properties, estimation of parameter and applicationshowing the superiority of Lindley distribution over exponential distribution has been done by Ghitany et al.(2008). Shanker et al. (2015) have comparative study on modeling of lifetime data using one parameter Lindley (1958) distribution and exponential distribution and concluded that there are many lifetime data where exponential distribution gives better fit than Lindley distribution.

Shanker and Mishra (2013) proposed the two parameter Lindley distribution,of which the one parameter Lindley distribution is a particular case, and have discussed its various properties and have shown that the two parameter Lindley distribution provides a better alternative to the one parameter Lindley distribution.

2. Mid-truncated distribution

We define the Mid-truncated distribution as follows:
Let Y be a continuous random variable with baseline probability density function (pdf) $g(y)$ and cumulative distribution function (cdf) $G(y)$. Define X as a corresponding mid-truncated variable, of the random variable Y, with pdf $f(x)$. We define

$$f(x) = \begin{cases} p\frac{g(x)}{G(Q_1)}, & if x \le Q_1 \\ q\frac{g(x)}{1-G(P_1)}, & if x \ge P_1 \end{cases}, \; 0 \le p \le 1,\; 0 \le q \le 1,\; s.t.\;\; p+q=1, Q_1 \le P_1, \tag{2.1}$$

which is called the mid-truncated density function, and Q_1 and P_1 are the points of mid truncation of the baseline distribution under consideration. Also we assume that

$$Q = \int_{-\infty}^{Q_1} g(x)dx, \tag{2.2}$$

and

$$1-P = \int_{P_1}^{\infty} g(x)dx. \tag{2.3}$$

Then equation (2.1) can be rewritten as

$$f(x) = \begin{cases} p\frac{g(x)}{Q}, & if x \le Q_1 \\ q\frac{g(x)}{1-P}, & if x \ge P_1 \end{cases}, \; 0 \le p \le 1,\; 0 \le q \le 1,\; s.t.\;\; p+q=1, Q_1 \le P_1. \tag{2.4}$$

The distribution function of a mid-truncated random variable X is given by:

$$F(x) = \begin{cases} p\frac{G(x)}{G(Q_1)}, & -\infty < x \le Q_1 \\ 1-q\frac{1-G(x)}{1-G(P_1)}, & P_1 \le x < \infty. \end{cases} \tag{2.5}$$

2.1 Statistical Measures

The ***k*-th moment** of any arbitrary mid – truncated random variable X defined over $(-\infty, Q_1) \cup (P_1, \infty)$, (denoted by $\mu_*^{(k)}$) is given by:

$$\mu_*^{(k)} = \frac{p}{G(Q_1)}\int_{-\infty}^{Q_1} x^k\, g(x)dx + \frac{q}{1-G(P_1)}\int_{P_1}^{\infty} x^k\, g(x)dx.$$

Integrating by parts, we get

$$\mu_*^{(k)} = \frac{p}{G(Q_1)}\left[{Q_1}^k G(Q_1) - k\int_{-\infty}^{Q_1} x^{k-1}\, G(x)dx\right] + \frac{q}{(1-G(P_1))}\left[\mu^{(k)} - {P_1}^k G(P_1) + k\int_{-\infty}^{P_1} x^{k-1}\, G(x)dx\right], \tag{2.6}$$

where $\mu^{(k)}$ is the k-th moment of the corresponding un-truncated random variable.
In a similar way, we can define **Moment generating function** ($M^*(t)$) and **Characteristic function** ($\Phi^*(t)$) of the random variable X as follows:

$$M^*(t) = pe^{tQ_1} - \frac{pt}{G(Q_1)}\int_{-\infty}^{Q_1} e^{xt}\, G(x)dx + \frac{q}{1-G(P_1)}[M(t) - e^{tP_1}G(P_1) + t\int_{-\infty}^{P_1} e^{tx}\, G(x)dx], \tag{2.7}$$

and

$$\Phi^*(t) = pe^{itQ_1} - \frac{pit}{G(Q_1)}\int_{-\infty}^{Q_1} e^{ixt}\, G(x)dx + \frac{q}{1-G(P_1)}[\Phi(t) - e^{itP_1}G(P_1) + it\int_{-\infty}^{P_1} e^{itx}\, G(x)dx], \tag{2.8}$$

where $M(t)$ and $\Phi(t)$ are the moment generating function and the characteristic function of the corresponding un-truncated random variable.

Note: In the following sections we will consider $p = q = \frac{1}{2}$.

3. Mid-truncated two-parameter Lindley distribution

Shanker and Mishra (2013) introduced a two parameter Lindley distribution with its probability density function (pdf) given as

$$g(x) = \frac{\lambda^2}{(1+\alpha\lambda)}(\alpha + x)e^{-\lambda x}, \quad x > 0,\ \lambda > 0, \alpha\lambda > -1, \tag{3.1}$$

and the cumulative distribution function(cdf) as

$$G(x) = 1-\left(\frac{1+\alpha\lambda+\lambda x}{1+\alpha\lambda}\right)e^{-\lambda x}, \quad x > 0,\ \lambda > 0, \alpha\lambda > -1. \tag{3.2}$$

Then the probability density function (pdf) of mid-truncated two-parameter Lindleydistribution is given by

$$f(x) = \begin{cases} \frac{\lambda^2(\alpha+x)e^{-\lambda x}}{2Q(1+\alpha\lambda)} & , \quad 0 < x \le Q_1 \\ \frac{\lambda^2(\alpha+x)e^{-\lambda x}}{2(1-P)(1+\alpha\lambda)} & , \quad P_1 \le x < \infty \end{cases} \tag{3.3}$$

and the cumulative distribution function(cdf) is given by

$$F(x) = \begin{cases} \frac{1}{2Q}(1 - (\frac{1+\alpha\lambda+\lambda x}{1+\alpha\lambda})e^{-\lambda x}) & , \quad 0 < x \le Q_1 \\ 1 - \frac{1}{2(1-P)}(\frac{1+\alpha\lambda+\lambda x}{1+\alpha\lambda})e^{-\lambda x} & , \quad P_1 \le x < \infty. \end{cases} \tag{3.4}$$

Using (3.3) and (3.4), we get the relation between pdf and cdf as

$$1 - F(x) = \begin{cases} 1 - \frac{1}{2Q} + f(x)\left(\frac{1+\alpha\lambda+\lambda x}{\lambda^2(\alpha+x)}\right) & , \quad 0 < x \le Q_1 \\ f(x)\left(\frac{1+\alpha\lambda+\lambda x}{\lambda^2(\alpha+x)}\right) & , \quad P_1 \le x < \infty. \end{cases} \tag{3.5}$$

The mid-truncated two-parameter Lindley density function for $\alpha = 1.5$ and $\lambda = 0.5$, $p = q = 0.5$ truncated at $Q_1 = 2.0$ and $P_1 = 3.0$ is provided in Figure 3.1.

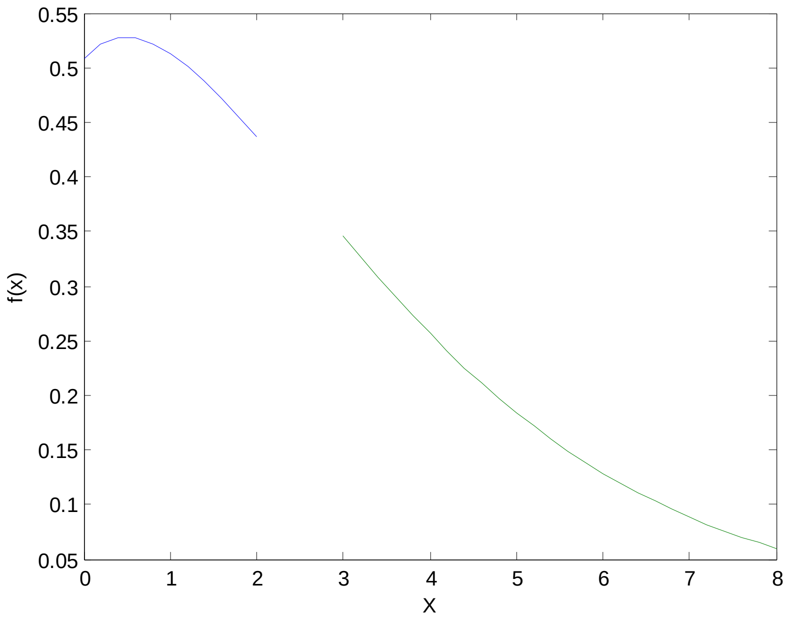

Figure 3.1: Probability density function for mid-truncated two-parameter Lindley distribution

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the mid-truncated two-parameter Lindleydistributiondefined in (3.3) and let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ be the corresponding order statistics. Thus the probability density function (pdf) of $X_{r:n}$ $(1 \le r \le n)$ is given by:

$$f_{r:n}(x) = c_{r:n}[F(x)]^{r-1}[1 - F(x)]^{n-r} f(x), \quad 0 \le x < \infty, \tag{3.6}$$

where $c_{r:n} = \frac{n!}{(r-1)!(n-r)!}$.

The joint density function of order statistics $X_{r:n}$ and $X_{s:n}$ $(1 \le r < s \le n)$ is given by

$$f_{r,s:n}(x,y) = c_{r,s:n}[F(x)]^{r-1}[F(y) - F(x)]^{s-r-1}[1 - F(y)]^{n-s} f(y) f(x), \quad 0 \le x < y < \infty,$$

where $c_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$. (3.7)

The single moments of order statistics $X_{r:n}$ $(1 \le r \le n)$ are given by

$$\mu_{r:n}^{(k)} = E(X_{r:n}^k) = \int_0^{Q_1} x^k f_{r:n}(x)dx + \int_{P_1}^{\infty} x^k f_{r:n}(x)dx, k = 0,1,2,\ldots\ldots \tag{3.8}$$

Similarly, the product moments of $X_{r:n}$ and $X_{s:n}$ $(1 \le r < s \le n)$, are given by

$$\mu_{r,s:n}^{(j,k)} = E(X_{r:n}^j X_{s:n}^k) = \int_0^{Q_1}\int_x^{Q_1} x^j y^k f_{r,s:n}(x,y)dydx + \int_{P_1}^{\infty}\int_x^{\infty} x^j y^k f_{r,s:n}(x,y)dydx, \tag{3.9}$$

$$j, k = 0,1,2,\ldots\ldots$$

3.1 K-th moment

The k-th moment of the mid – truncated two-parameter Lindley distribution can be obtained using(2.6) and is given by:

$$\mu_*^{(k)} = \frac{1}{2G(Q_1)}\left[{Q_1}^k G(Q_1) - k\int_0^{Q_1} x^{k-1}\, G(x)dx\right] + \frac{1}{2(1-G(P_1))}\left[\mu^{(k)} - {P_1}^k G(P_1) + k\int_0^{P_1} x^{k-1}\, G(x\,)dx\right]$$

$$= \frac{Q_1^k}{2} + \frac{1}{2(1-G(P_1))}\left[\mu^{(k)} - {P_1}^k G(P_1)\right] - \frac{k}{2G(Q_1)}\int_0^{Q_1} x^{k-1}\, G(x)dx + \frac{k}{2(1-G(P_1))}\int_0^{P_1} x^{k-1}\, G(x)dx.$$

Substituting the value of G(x) from equation (3.2) and solving, we get

$$\mu_*^{(k)} = \frac{Q_1^k}{2} + \frac{1}{2(1-G(P_1))}\left[\mu^{(k)} - {P_1}^k G(P_1)\right]$$

$$-\frac{k}{2G(Q_1)}\left[\frac{Q_1^k}{k} - \frac{1}{\lambda^k}\left\{\gamma(k,\lambda Q_1) + \frac{\gamma(k+1,\lambda Q_1)}{(1+\alpha\lambda)}\right\}\right] + \frac{k}{2(1-G(P_1))}\left[\frac{P_1^k}{k} - \frac{1}{\lambda^k}\left\{\gamma(k,\lambda P_1) + \frac{\gamma(k+1,\lambda P_1)}{(1+\alpha\lambda)}\right\}\right], \quad (3.10)$$

where $\gamma(a\,,\,b\,) = \int_0^b e^{-x}\, x^{a-1} dx$, is the lower incomplete gamma function and can be obtained using tables given by Pearson(1965).

To investigate the effect of the parameters α andλon the density function of the mid-truncated two-parameter Lindleydistribution,we have computedmeans, variances, skewness and kurtosis for different values of the parameters, takingQ_1= 2.0 and P_1= 3.0 ,which are presented in following Tables 3.1 and 3.2.

Table 3.1

α= 3.5,$Q_1 = 2.0$,$P_1 = 3.0$

λ	Mean	Variance	Skewness(β_1)	Kurtosis(β_2)
1	2.4431	3.6289	0.3037	2.8076
1.5	2.1728	2.8057	0.0517	1.8701
2	2.0212	2.5338	0.0099	1.4955
2.5	1.9219	2.418	0.0022	1.3165
3	1.8522	2.3603	6.03E-04	1.2185

Table 3.2

λ= 2.5,$Q_1 = 2.0$,$P_1 = 3.0$

Α	Mean	Variance	Skewness(β_1)	Kurtosis(β_2)
1	1.9592	2.3637	0.0017	1.3546
2	1.9358	2.4029	0.0021	1.3314
4	1.9192	2.42	0.0022	1.3135
7.5	1.909	2.4237	0.0022	1.3015
10	1.9056	2.4236	0.0022	1.2972

3.2 Moment generating function and characteristic function

The moment generating function of themid – truncated two-parameter Lindley distribution canbe obtained using (2.7) and is given by:

$$M^*(t) = \frac{e^{tQ_1}}{2} - \frac{t}{2G(Q_1)}\int_0^{Q_1} e^{xt}\, G(x)\, dx + \frac{1}{2(1-G(P_1))}\left[M(t) - e^{tP_1}G(P_1) + t\int_0^{P_1} e^{tx}\, G(x)\, dx\right]$$

$$= \frac{e^{tQ_1}}{2} + \frac{1}{2(1-G(P_1))}\left[M(t) - e^{tP_1}G(P_1)\right] - \frac{t}{2G(Q_1)}\int_0^{Q_1} e^{xt}\, G(x)dx + \frac{t}{2(1-G(P_1))}\int_0^{P_1} e^{tx}\, G(x)dx.$$

Substituting the value of $G(x) = 1 - \left(\frac{1+\alpha\lambda+\lambda x}{1+\alpha\lambda}\right)e^{-\lambda x}$ from (3.2), and solving we get

$$M^*(t) = \frac{e^{tQ_1}}{2} + \frac{1}{2(1-G(P_1))}\left[M(t) - e^{tP_1}G(P_1)\right] - \frac{1}{2G(Q_1)}\left[(e^{tQ_1} - 1) - \frac{t}{t-\lambda}(e^{-Q_1(\lambda-t)} - 1) - \frac{\lambda t}{(\lambda-t)^2(1+\alpha\lambda)}\gamma(2,(\lambda-t)Q_1)\right]$$

$$+ \frac{1}{2(1-G(P_1))}\left[(e^{tP_1} - 1) - \frac{t}{t-\lambda}(e^{-P_1(\lambda-t)} - 1) - \frac{\lambda t}{(\lambda-t)^2(1+\alpha\lambda)}\gamma(2,(\lambda-t)P_1)\right], \quad (3.11)$$

where $\gamma(a\,,\,b\,) = \int_0^b e^{-x}\, x^{a-1} dx$, is the lower incomplete gamma function and can be obtained using tables given by Pearson (1965).

Similarly, the characteristic function of the mid-truncated two-parameter Lindley distribution can be obtained.

4. Recurrence relations for single moments of order statistics from mid-truncated two - parameter Lindley distribution

In this section we shall derive recurrence relations for single moments of order statistics frommid-truncated two-parameter Lindleydistribution.

Lemma 4.1: In usual notations, for amid-truncated distribution,we have, for k =1,2,3,…and1<$r \le n$,

$$\mu_{r:n}^{(k)} - \mu_{r-1:n-1}^{(k)} = \binom{n-1}{r-1}[k\int_0^{Q_1} x^{k-1}\left(F(x)\right)^{r-1}\left(1 - F(x)\right)^{n-r+1} dx$$

$$+ k\int_{P_1}^{\infty} x^{k-1}\big(F(x)\big)^{r-1}\big(1-F(x)\big)^{n-r+1}dx - \left(\tfrac{1}{2}\right)^n ({Q_1}^k - {P_1}^k)\ ]. \tag{4.1}$$

Proof: Using (3.8), we get

$$\mu_{r:n}^{(k)} - \mu_{r-1:n-1}^{(k)} = C_{r:n}\int_0^{Q_1} x^k F(x)^{r-1}\big(1-F(x)\big)^{n-r} f(x)\,dx + C_{r:n}\int_{P_1}^{\infty} x^k F(x)^{r-1}\big(1-F(x)\big)^{n-r} f(x)\,dx$$

$$-C_{r-1:n-1}\int_0^{Q_1} x^k F(x)^{r-2}\big(1-F(x)\big)^{n-r} f(x)\,dx - C_{r-1:n-1}\int_{P_1}^{\infty} x^k F(x)^{r-2}\big(1-F(x)\big)^{n-r} f(x)\,dx$$

$$= \binom{n-1}{r-1}\Big[\int_0^{Q_1} x^k F(x)^{r-2}\big(1-F(x)\big)^{n-r} f(x)\{nF(x)-(r-1)\}\,dx$$
$$+ \int_{P_1}^{\infty} x^k F(x)^{r-2}\big(1-F(x)\big)^{n-r} f(x)\{nF(x)-(r-1)\}\,dx\Big]. \tag{4.2}$$

Let $\varphi(x) = -(F(x))^{r-1}\big(1-F(x)\big)^{n-r+1}$. (4.3)

Differentiating both sides with respect to x, we get

$$\frac{d\varphi(x)}{dx} = (F(x))^{r-2}\big(1-F(x)\big)^{n-r} f(x)\{nF(x)-(r-1)\}. \tag{4.4}$$

On using (4.4), equation (4.2) becomes

$$\mu_{r:n}^{(k)} - \mu_{r-1:n-1}^{(k)} = \binom{n-1}{r-1}\left[\int_0^{Q_1} x^k \left(\frac{d\varphi(x)}{dx}\right)dx + \int_{P_1}^{\infty} x^k \left(\frac{d\varphi(x)}{dx}\right)dx\right]$$
$$= \binom{n-1}{r-1}\left[[x^k\varphi(x)]_0^{Q_1} - k\int_0^{Q_1} x^{k-1}\varphi(x)dx + [x^k\varphi(x)]_{P_1}^{\infty} - k\int_{P_1}^{\infty} x^{k-1}\varphi(x)dx\right]$$
$$= \binom{n-1}{r-1}\left[{Q_1}^k\varphi(Q_1) - {P_1}^k\varphi(P_1) - k\int_0^{Q_1} x^{k-1}\varphi(x)dx - k\int_{P_1}^{\infty} x^{k-1}\varphi(x)dx\right]. \tag{4.5}$$

Now, since $F(Q_1) = \frac{1}{2}$ and $F(P_1) = \frac{1}{2}$, therefore, from (4.3), we have $\varphi(Q_1) = \varphi(P_1) = -(\frac{1}{2})^n$.

Substituting the values of $\varphi(x), \varphi(Q_1)$ and $\varphi(P_1)$ in (4.5), it leads to (4.1).

Lemma 4.2: If, for $1 \le r \le n$ and $k = 1,2,...,$

$$I_{r:n}^{(k)} = \binom{n-1}{r-1}\int_0^{Q_1} x^{k-1}\,(F(x))^{r-1}(1-F(x))^{n-r}dx, \tag{4.6}$$

then, it may be written as

$$I_{1:n}^{(k)} = \begin{cases} \dfrac{Q_1^k}{k}, & n = 1 \\ \sum_{j=0}^{n-1}\sum_{i=0}^{j}\sum_{t=0}^{i}(-1)^{i+j}\left(\frac{1}{2Q}\right)^j \binom{n-1}{j}\binom{j}{i}\binom{i}{t}\left(\frac{\lambda}{\alpha\lambda+1}\right)^t \frac{\gamma(k+t,\lambda i Q_1)}{(\lambda i)^{k+t}}, & n > 1, \end{cases} \tag{4.7}$$

and

$$I_{r:n}^{(k)} = \begin{cases} \displaystyle\sum_{j=r}^{n}\sum_{i=0}^{j-1}\sum_{t=0}^{i}(-1)^{i+j-r}\left(\frac{1}{2Q}\right)^{j-1}\binom{n-1}{j-1}\binom{j-1}{r-1}\binom{j-1}{i}\binom{i}{t}\left(\frac{\lambda}{\alpha\lambda+1}\right)^t \frac{\gamma(k+t,\lambda i Q_1)}{(\lambda i)^{k+t}}, & 2 \le r < n \\ \displaystyle\sum_{i=0}^{n-1}\sum_{t=0}^{i}(-1)^i\left(\frac{1}{2Q}\right)^{n-1}\binom{n-1}{i}\binom{i}{t}\left(\frac{\lambda}{\alpha\lambda+1}\right)^t \frac{\gamma(k+t,\lambda i Q_1)}{(\lambda i)^{k+t}}, & r = n, \end{cases} \tag{4.8}$$

where $\gamma(a, b) = \int_0^b e^{-x}x^{a-1}dx$, is the lower incomplete gamma function and can be obtained using tables given by Pearson (1965).

Proof: Relations in (4.7) and (4.8) (for $r = n$) may be proved by following exactly the same steps as those in proving (4.8)(for $2 \le r < n$), which is presented below.

Expanding $(1-F(x))^{n-r}$ binomially in powers of $F(x)$ and then substituting the same in (4.6), we get for $2 \le r < n$,

$$I_{r:n}^{(k)} = \binom{n-1}{r-1}\sum_{l=0}^{n-r}(-1)^l\binom{n-r}{l}\int_0^{Q_1} x^{k-1}\,(F(x))^{l+r-1}dx.$$
$$= \binom{n-1}{r-1}\sum_{j=r}^{n}(-1)^{j-r}\binom{n-r}{j-r}\int_0^{Q_1} x^{k-1}\,(F(x))^{j-1}dx.$$

Substituting the value of $F(x)$ from (3.4), we get

$$I_{r:n}^{(k)} = \sum_{j=r}^{n}(-1)^{j-r}\binom{n-1}{j-1}\binom{j-1}{r-1}\frac{1}{(2Q)^{j-1}}\int_0^{Q_1} x^{k-1}\left(1-\left(\frac{1+\alpha\lambda+\lambda x}{1+\alpha\lambda}\right)e^{-\lambda x}\right)^{j-1}dx.$$

Expanding $\left(1-\left(\frac{1+\alpha\lambda+\lambda x}{1+\alpha\lambda}\right)e^{-\lambda x}\right)^{j-1}$ binomially, we get

$$I_{r:n}^{(k)} = \sum_{j=r}^{n} \sum_{i=1}^{j-1} (-1)^{j-r+i} \left(\frac{1}{2Q}\right)^{j-1} \binom{n-1}{j-1} \binom{j-1}{r-1} \binom{j-1}{i} \int_0^{Q_1} x^{k-1} \left(\frac{1+\alpha\lambda+\lambda x}{1+\alpha\lambda}\right)^i e^{-\lambda i x} dx. \quad (4.9)$$

Further, expanding $\left(\frac{1+\alpha\lambda+\lambda x}{1+\alpha\lambda}\right)^i = \left(1+\frac{\lambda x}{1+\alpha\lambda}\right)^i$ binomially and solving the integral, we get the desired result (4.8) for the case $2 \le r < n$.

Theorem 4.1: For $k > 0, \lambda > 0, \alpha\lambda > -1$, we have

$$\mu_{1:n}^{(k+1)} = \frac{k+1}{k}\left(\frac{k}{n\lambda} - \alpha\right)\mu_{1:n}^{(k)} + \frac{(k+1)(1+\lambda\alpha)}{n\lambda^2}\mu_{1:n}^{(k-1)} + (k+1)\left(\frac{2Q-1}{Q}\right)\left[\alpha I_{1:n}^{(k)} + I_{1:n}^{(k+1)}\right]$$
$$-\frac{k+1}{k}\frac{1}{2^n}\left[\alpha\left({Q_1}^k - {P_1}^k\right) + \frac{k}{k+1}\left({Q_1}^{k+1} - {P_1}^{k+1}\right)\right], \quad (4.10)$$

where the values of $I_{1:n}^{(k)}$ are given in (4.7).

Proof: Relation in (4.10) may be proved by following exactly the same steps as those in proving Theorem 4.2 which is presented next.

Theorem 4.2: For $1 < r \le n$, $k > 0$, $\lambda > 0, \alpha\lambda > -1$, we have

$$\mu_{r:n}^{(k+1)} = \frac{k+1}{k}\left(\frac{k}{n\lambda} - \alpha\right)\mu_{r:n}^{(k)} + \frac{(k+1)(1+\lambda\alpha)}{n\lambda^2}\mu_{r:n}^{(k-1)} + \mu_{r-1:n-1}^{(k+1)} + \frac{k+1}{k}\alpha\mu_{r-1:n-1}^{(k)}$$
$$+(k+1)\left(\frac{2Q-1}{Q}\right)\left[\alpha I_{r:n}^{(k)} + I_{r:n}^{(k+1)}\right] - \frac{k+1}{k}\binom{n-1}{r-1}\frac{1}{2^n}\left[\alpha\left({Q_1}^k - {P_1}^k\right) + \frac{k}{k+1}\left({Q_1}^{k+1} - {P_1}^{k+1}\right)\right], \quad (4.11)$$

where the values of $I_{r:n}^{(k)}$, for different values of r, are given in (4.8).

Proof: Using (3.5) in (4.1), consider

$$\alpha\left(\mu_{r:n}^{(k)} - \mu_{r-1:n-1}^{(k)}\right) + \frac{k}{k+1}\left(\mu_{r:n}^{(k+1)} - \mu_{r-1:n-1}^{(k+1)}\right)$$
$$= \binom{n-1}{r-1}\Big[k\int_0^{Q_1} x^{k-1}\left(F(x)\right)^{r-1}\left(1-F(x)\right)^{n-r}(\alpha+x) \times \left\{\left(\frac{2Q-1}{2Q}\right) + \frac{1+\lambda(\alpha+x)}{\lambda^2(\alpha+x)}f(x)\right\}dx$$
$$+k\int_{P_1}^{\infty} x^{k-1}\left(F(x)\right)^{r-1}\left(1-F(x)\right)^{n-r}(\alpha+x)\left\{\frac{1+\lambda(\alpha+x)}{\lambda^2(\alpha+x)}f(x)\right\}dx$$
$$-\frac{1}{2^n}\left\{\alpha\left({Q_1}^k - {P_1}^k\right) + \frac{k}{k+1}\left({Q_1}^{k+1} - {P_1}^{k+1}\right)\right\}\Big]$$

$$= \frac{k(1+\lambda\alpha)}{n\lambda^2}\mu_{r:n}^{(k-1)} + \frac{k}{n\lambda}\mu_{r:n}^{(k)} + k\left(\frac{2Q-1}{2Q}\right)\binom{n-1}{r-1}\left[\int_0^{Q_1} x^{k-1}\left(F(x)\right)^{r-1}\left(1-F(x)\right)^{n-r}(\alpha+x)dx\right]$$
$$-\binom{n-1}{r-1}\frac{1}{2^n}\left[\alpha\left({Q_1}^k - {P_1}^k\right) + \frac{k}{k+1}\left({Q_1}^{k+1} - {P_1}^{k+1}\right)\right]$$
$$= \frac{k(1+\lambda\alpha)}{n\lambda^2}\mu_{r:n}^{(k-1)} + \frac{k}{n\lambda}\mu_{r:n}^{(k)} + k\left(\frac{2Q-1}{2Q}\right)\left[\alpha I_{r:n}^{(k)} + I_{r:n}^{(k+1)}\right] - \binom{n-1}{r-1}\frac{1}{2^n}\left[\alpha\left({Q_1}^k - {P_1}^k\right) + \frac{k}{k+1}\left({Q_1}^{k+1} - {P_1}^{k+1}\right)\right].$$

On simplification, we get

$$\left(\alpha - \frac{k}{n\lambda}\right)\mu_{r:n}^{(k)} - \alpha\mu_{r-1:n-1}^{(k)} + \frac{k}{k+1}\left(\mu_{r:n}^{(k+1)} - \mu_{r-1:n-1}^{(k+1)}\right)$$
$$= \frac{k(1+\lambda\alpha)}{n\lambda^2}\mu_{r:n}^{(k-1)} + k\left(\frac{2Q-1}{2Q}\right)\left[\alpha I_{r:n}^{(k)} + I_{r:n}^{(k+1)}\right] - \binom{n-1}{r-1}\frac{1}{2^n}\left[\alpha\left({Q_1}^k - {P_1}^k\right) + \frac{k}{k+1}\left({Q_1}^{k+1} - {P_1}^{k+1}\right)\right], \quad (4.12)$$

where the values of $I_{r:n}^{(k)}$, for different values of r, are given in (4.8). Rearrangement of terms in (4.12) leads to the desired result in (4.11).

5. Recurrence relations for product moments of order statistics from mid-truncated two-parameter Lindley distribution

Lemma 5.1: For $1 \le r < s \le n$, and $j, k > 0$, we have

$$\mu_{r,s:n}^{(j,k)} - \mu_{r,s-1:n}^{(j,k)} = C_{r,s:n}^{*}\Big[k\int_0^{Q_1}\int_x^{Q_1} x^j y^{k-1}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s+1}f(x)dydx$$
$$+ k\int_{P_1}^{\infty}\int_x^{\infty} x^j y^{k-1}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s+1}f(x)dydx$$
$$-{Q_1}^k\left(\frac{1}{2}\right)^{n-r}\sum_{i=0}^{s-r-1}\binom{s-r-1}{i}(-1)^i \times \int_0^{Q_1} x^j\,[F(x)]^{r+i-1}[1-F(x)]^{s-r-i-1}f(x)dx\Big], \quad (5.1)$$

where $C_{r,s:n}^{*} = \frac{n!}{(r-1)!(s-r-1)!(n-s+1)!}$.

Proof: Using equation (3.9), we have

$$\mu_{r,s:n}^{(j,k)} - \mu_{r,s-1:n}^{(j,k)} =$$
$$\frac{n!}{(r-1)!(s-r-1)!(n-s)!} \times \Big[\int_0^{Q_1}\int_x^{Q_1} x^j\,y^k[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(x)f(y)dydx$$
$$+\int_{P_1}^{\infty}\int_x^{\infty} x^j\,y^k[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(x)f(y)dydx\Big]$$
$$-\frac{n!}{(r-1)!(s-r-2)!(n-s+1)!} \times \Big[\int_0^{Q_1}\int_x^{Q_1} x^j\,y^k[F(x)]^{r-1}[F(y)-F(x)]^{s-r-2}[1-F(y)]^{n-s+1}f(x)f(y)dydx$$

$+\int_{P_1}^{\infty}\int_{x}^{\infty} x^j\, y^k [F(x)]^{r-1}[F(y)-F(x)]^{s-r-2}[1-F(y)]^{n-s+1} f(x)f(y)dydx]$

$=C^*_{r,s:n}[\int_0^{Q_1}\int_x^{Q_1} x^j\, y^k [F(x)]^{r-1}[F(y)-F(x)]^{s-r-2}[1-F(y)]^{n-s} f(x)f(y)$

$\times\{(n-s+1)\big(F(y)-F(x)\big)-(s-r-1)\big(1-F(y)\big)\}dydx$

$+\int_{P_1}^{\infty}\int_{x}^{\infty} x^j\, y^k [F(x)]^{r-1}[F(y)-F(x)]^{s-r-2}[1-F(y)]^{n-s} f(x)f(y)$

$\times\{(n-s+1)(F(y)-F(x))-(s-r-1)(1-F(y))\}dydx]$

$=C^*_{r,s:n}[\int_0^{Q_1}\int_x^{Q_1} x^j\, y^k [F(x)]^{r-1}[F(y)-F(x)]^{s-r-2}[1-F(y)]^{n-s} f(x)f(y)$

$\times\{(n-r)F(y)-(n-s+1)F(x)-(s-r-1)\}dydx$

$$+\int_{P_1}^{\infty}\int_{x}^{\infty} x^j\, y^k [F(x)]^{r-1}[F(y)-F(x)]^{s-r-2}[1-F(y)]^{n-s} f(x)f(y)$$

$\times\{(n-r)F(y)-(n-s+1)F(x)-(s-r-1)\}dydx].$ (5.2)

Let

$\varphi(x,y)=-[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s+1}$. (5.3)

Then,

$\frac{\partial\varphi(x,y)}{\partial y}=-(s-r-1)[F(y)-F(x)]^{s-r-2}[1-F(y)]^{n-s+1}f(y)$

$+(n-s+1)[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(y)$

$=[F(y)-F(x)]^{s-r-2}[1-F(y)]^{n-s}f(y)[(n-r)F(y)-(n-s+1)F(x)-(s-r-1)].$

(5.4)

Putting the above value in (5.2), we get

$\mu^{(j,k)}_{r,s:n}-\mu^{(j,k)}_{r,s-1:n}=C^*_{r,s:n}[\int_0^{Q_1}\int_x^{Q_1} x^j\, y^k [F(x)]^{r-1}\frac{\partial\varphi(x,y)}{\partial y}f(x)dydx+\int_{P_1}^{\infty}\int_x^{\infty} x^j\, y^k [F(x)]^{r-1}\frac{\partial\varphi(x,y)}{\partial y}f(x)dydx]$

$=C^*_{r,s:n}[\int_0^{Q_1} x^j [F(x)]^{r-1}\left(\int_x^{Q_1} y^k \frac{\partial\varphi(x,y)}{\partial y}\,\mathrm{dy}\right)f(x)dx+\int_{P_1}^{\infty} x^j [F(x)]^{r-1}\left(\int_x^{\infty} y^k \frac{\partial\varphi(x,y)}{\partial y}\,\mathrm{dy}\right)f(x)dx].$ (5.5)

Now, consider $\int_x^{Q_1} y^k \frac{\partial\varphi(x,y)}{\partial y}dy$

$=Q_1{}^k\varphi(x,Q_1)+k\int_x^{Q_1} y^{k-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s+1}dy.$ (5.6)

From equations(5.3) and (3.4), we have

$\varphi(x,Q_1)=-\left(\frac{1}{2}-F(x)\right)^{s-r-1}\left(\frac{1}{2}\right)^{n-s+1}.$ (5.7)

Putting (5.7) in equation (5.6), we get

$\int_x^{Q_1} y^k \frac{\partial\varphi(x,y)}{\partial y}dy=k\int_x^{Q_1} y^{k-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s+1}dy$

$-Q_1{}^k[1-2F(x)]^{s-r-1}\left(\frac{1}{2}\right)^{n-r}.$ (5.8)

Also,

$\int_x^{\infty} y^k \frac{\partial\varphi(x,y)}{\partial y}dy=k\int_x^{\infty} y^{k-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s+1}dy.$ (5.9)

Putting (5.8) and (5.9) in equation (5.5), we get

$\mu^{(j,k)}_{r,s:n}-\mu^{(j,k)}_{r,s-1:n}=C^*_{r,s:n}[\int_0^{Q_1} x^j [F(x)]^{r-1}\{k\int_x^{Q_1} y^{k-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s+1}dy$

$-Q_1{}^k[1-2F(x)]^{s-r-1}\left(\frac{1}{2}\right)^{n-r}\}f(x)dx$

$+\int_{P_1}^{\infty} x^j [F(x)]^{r-1}\{k\int_x^{\infty} y^{k-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s+1}dy\}f(x)dx]$

$=C^*_{r,s:n}[k\int_0^{Q_1}\int_x^{Q_1} x^j y^{k-1}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s+1}f(x)dydx$

$+k\int_{P_1}^{\infty}\int_x^{\infty} x^j y^{k-1}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s+1}f(x)dydx$

$-\left(\frac{1}{2}\right)^{n-r}\int_0^{Q_1} x^j Q_1{}^k[1-2F(x)]^{s-r-1}[F(x)]^{r-1}f(x)dx].$ (5.10)

Considering the last term of the above equation, and expanding $((1-F(x))-F(x))^{s-r-1}$binomially and simplifying, we get the desired result (5.1).

Lemma 5.2: If for j > 1

$I_j(a,b)=\int_0^{Q_1} x^j [F(x)]^a[1-F(x)]^b f(x)dx,$ (5.11)

where *f(x)* and *F(x)* are the pdf and cdf, respectively, of a general mid-truncated distribution, then, it can be rewritten as

$I_j(a,b)=\sum_{w=0}^{b}\binom{b}{w}(-1)^w\int_0^{Q_1} x^j [F(x)]^{a+w} f(x)dx$

$= \sum_{w=0}^{b}\binom{b}{w}(-1)^w\, I_j(a+w,0).$ (5.12)

Proof:Considering R.H.S of (5.11) and expanding $[1-F(x)]^b$binomially we get the desired result.

Lemma 5.3: Formid-truncated two-parameter Lindley distribution defined in (3.3),$I_j(a,0)$ can be written as

$$I_j(a,0)=\left(\frac{1}{2Q}\right)^{a+1}\sum_{g=0}^{a}\binom{a}{g}(-1)^g\sum_{h=0}^{g}\binom{g}{h}\frac{\lambda^{1-j}}{(1+\alpha\lambda)^{h+1}}\frac{1}{(1+g)^{j+h+1}}$$
$$\times\left[\alpha\gamma\big(j+h+1,\lambda Q_1(1+g)\big)+\frac{\gamma(j+h+2,\lambda Q_1(1+g))}{\lambda(1+g)}\right], \quad (5.13)$$

where$\gamma(a\,,b)=\int_0^b e^{-x}\,x^{a-1}dx$, is the lower incomplete gamma function and can be obtained using tables given byPearson (1965).

Proof: Using (5.12) for b = 0 and substituting in it the values of *f(x)* and *F(x)* from (3.3) and (3.4), respectively for mid-truncated two-parameter Lindley distribution and simplifying,we get the desired result (5.13).

Lemma 5.4: If for *1 ≤ r < s ≤ n and j, k > 0,*

$$I_{r,s:n}^{(j,k)}=\int_0^{Q_1}\int_x^{Q_1}x^j y^{k-1}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(x)dydx, \quad (5.14)$$

where *f(x)* and *F(x)* are pdf and cdf respectively, of a general mid-truncated distribution,
then it can be rewritten as

$$I_{r,s:n}^{(j,k)}=\sum_{l=0}^{s-r-1}\binom{s-r-1}{l}\sum_{m=0}^{n-s}\binom{n-s}{m}(-1)^{s-r-l+m-1}\int_0^{Q_1}\int_x^{Q_1}x^j y^{k-1}[F(x)]^{s-l-2}[F(y)]^{l+m}dydx$$
$$=\sum_{l=0}^{s-r-1}\binom{s-r-1}{l}\sum_{m=0}^{n-s}\binom{n-s}{m}(-1)^{s-r-l+m-1}\,T_{j,k-1}(s-l-2,l+m), \quad (5.15)$$

where

$$T_{u,v}(a,b)=\int_0^{Q_1}\int_x^{Q_1}x^u y^v\,[F(x)]^a[F(y)]^b f(x)dydx. \quad (5.16)$$

Further, for mid-truncated two-parameter Lindley distribution defined in (3.3),$T_{u,v}(a,b)$ can be simplified as

$$T_{u,v}(a,b)=\left(\frac{1}{2Q}\right)^b\sum_{c=0}^{b}\sum_{d=0}^{c}(-1)^c\binom{b}{c}\binom{c}{d}\left(\frac{\lambda}{1+\alpha\lambda}\right)^d\frac{1}{(\lambda c)^{v+d+1}}$$
$$\times[(\gamma(v+d+1,\lambda cQ_1)-(v+d)!)I_u(a,0)$$
$$+(v+d)!\sum_{t=0}^{v+d}\frac{(\lambda c)^t}{t!}\frac{\lambda^2}{(1+\alpha\lambda)}\left(\frac{1}{2Q}\right)^a\sum_{f=0}^{a}\binom{a}{f}(-1)^f\sum_{e=0}^{f}\binom{f}{e}\left(\frac{\lambda}{1+\alpha\lambda}\right)^e$$
$$\times\frac{1}{(\lambda(a+c+1))^{u+t+e+1}}\left\{\alpha\gamma\big(u+t+e+1,\lambda Q_1(a+c+1)\big)+\frac{\gamma(u+t+e+2,\lambda Q_1(a+c+1))}{\lambda(a+c+1)}\right\}], \quad (5.17)$$

where $I_u(a,0)$ is defined in (5.13) and $\gamma(a\,,b)=\int_0^b e^{-x}\,x^{a-1}dx$, is the lower incomplete gamma function.

Proof: Substituting the values of *f(x)* and *F(x)* from (3.3) and (3.4) in (5.16) and solving the integral we get the desired expression in (5.17).

Theorem 5.1: For *1 ≤ r < s ≤ n and j, k > 0*,we have for mid-truncated two-parameterLindley distribution

$$\mu_{r,s:n}^{(j,k+1)}=\left(\frac{k+1}{k}\right)\mu_{r,s:n}^{(j,k)}\left(\frac{k}{\lambda(n-s+1)}-\alpha\right)+\frac{(k+1)(1+\alpha\lambda)}{\lambda^2(n-s+1)}\mu_{r,s:n}^{(j,k-1)}+\mu_{r,s-1:n}^{(j,k+1)}+\alpha\frac{k+1}{k}\mu_{r,s-1:n}^{(j,k)}$$
$$+C_{r,s:n}^{*}[k\left(\frac{2Q-1}{2Q}\right)\left(\alpha I_{r,s:n}^{(j,k)}+I_{r,s:n}^{(j,k+1)}\right)$$
$$-\left(\frac{1}{2}\right)^{n-r}Q_1{}^k\left(\alpha+\frac{kQ_1}{k+1}\right)\sum_{i=0}^{s-r-1}\binom{s-r-1}{i}(-1)^i I_j(r+i-1,s-r-i-1)], \quad (5.18)$$

where $C_{r,s:n}^{*}=\frac{n!}{(r-1)!(s-r-1)!(n-s+1)!}$, $I_j(a,b)$ for arbitrary a and b is evaluated using Lemma 5.3 and $I_{r,s:n}^{(j,k)}$is evaluated in Lemma 5.4.

Proof : Using (5.1),we can write

$$\alpha\left(\mu_{r,s:n}^{(j,k)}-\mu_{r,s-1:n}^{(j,k)}\right)+\frac{k}{k+1}\left(\mu_{r,s:n}^{(j,k+1)}-\mu_{r,s-1:n}^{(j,k+1)}\right)$$
$$=C_{r,s:n}^{*}[k\int_0^{Q_1}\int_x^{Q_1}x^j y^{k-1}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s+1}(\alpha+y)f(x)dydx$$

$$+k\int_{P_1}^{\infty}\int_x^{\infty}x^j y^{k-1}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s+1}(\alpha+y)f(x)dydx$$
$$-Q_1{}^k\left(\frac{1}{2}\right)^{n-r}\sum_{i=0}^{s-r-1}\binom{s-r-1}{i}(-1)^i\left(\alpha+\frac{k}{k+1}Q_1\right)\int_0^{Q_1}x^j\,[F(x)]^{r+i-1}[1-F(x)]^{s-r-i-1}f(x)dx]. \quad (5.19)$$

Using (3.5) in (5.19), we get

$$\alpha\left(\mu_{r,s:n}^{(j,k)}-\mu_{r,s-1:n}^{(j,k)}\right)+\frac{k}{k+1}\left(\mu_{r,s:n}^{(j,k+1)}-\mu_{r,s-1:n}^{(j,k+1)}\right)$$
$$=C_{r,s:n}^{*}[k\int_0^{Q_1}\int_x^{Q_1}x^j y^{k-1}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}\left\{\frac{2Q-1}{2Q}+\left(\frac{1+\alpha\lambda+\lambda y}{\lambda^2(\alpha+y)}\right)f(y)\right\}$$

$$\times(\alpha+y)f(x)dydx$$
$$+k\int_{P_1}^{\infty}\int_{x}^{\infty}x^j y^{k-1}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}\left\{f(x)\left(\frac{1+\alpha\lambda+\lambda x}{\lambda^2(\alpha+x)}\right)\right\}(\alpha+y)f(x)dydx$$
$$-{Q_1}^k\left(\frac{1}{2}\right)^{n-r}\sum_{i=0}^{s-r-1}\binom{s-r-1}{i}(-1)^i\left(\alpha+\frac{k}{k+1}Q_1\right)\int_0^{Q_1}x^j\,[F(x)]^{r+i-1}[1-F(x)]^{s-r-i-1}f(x)dx]$$
$$=\frac{k(1+\alpha\lambda)}{\lambda^2(n-s+1)}\mu_{r,s:n}^{(j,k-1)}+\frac{k}{\lambda(n-s+1)}\mu_{r,s:n}^{(j,k)}$$
$$+C_{r,s:n}^{*}[k\left(\frac{2Q-1}{2Q}\right)\int_0^{Q_1}\int_x^{Q_1}x^j y^{k-1}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}(\alpha+y)f(x)dydx$$
$$-\left(\frac{1}{2}\right)^{n-r}{Q_1}^k\left(\alpha+\frac{kQ_1}{k+1}\right)\sum_{i=0}^{s-r-1}\binom{s-r-1}{i}(-1)^i\times\int_0^{Q_1}x^j\,[F(x)]^{r+i-1}[1-F(x)]^{s-r-i-1}f(x)dx]. \tag{5.20}$$

Rearrangement of terms in (5.20) leads to the desired result in (5.18).

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