E-cordial labeling of Path union of wheel related graphs

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1. Abstract: In this paper we discuss e-cordial labeling of path union of wheel related graphs, Pm(G) where G is Wn, Gc, Ws, Gs, flag of (Wn) given by Fl(Wn), Fl(Ws), Fl(Gc), Fl(Gs) and show that under certain condition they are e- cordial. We also show that the e-cordial labeling of path union is independent of vertex on G used to form the path union.

Keywords: E-cordial, path union, fusion, edge, vertex, wheel. Gear graph.

Subject Classification: 05C78

2. Introduction

The graphs we discuss are simple, connected and finite. For terminology and definitions we depend on Graph Theory by Harary[4], Dynamic survey of graph labeling [3], and West [6].

In 1993 Yilmaz and Cahit [5] introduced a weaker version of edge graceful labeling called E-cordial. The word cordial was used first time in this paper. Let G be a graph with vertex set V and edge set E. Let f be a function that maps E into {0,1}. Define f on V by f(v) = \sum (f(uv)/2)E)(mod 2). The function f is called as E cordial labeling if |v(i)-v(i)|≤1 and |v(i)-v(i)|≤1. Where e(i) is the number of edges labeled with i = 0,1 and v(i) is the number of vertices labeled with i = 0,1. We also use v(i) = (a,b) to denote the number of vertices labeled with 0 are a in number and that with 1 are b in number. Similarly e(i) = (x,y) to denote number of edges labeled with 0 are x in number and that labeled with 1 are y in number. A lot of work has been done in this type of labeling and the above mentioned paper gave rise to number of papers on cordial labeling. A graph that admits E-cordial labeling is called as E- cordial graph. Yilmaz and Cahit has shown that Trees Tn with n vertices and Complete graphs Kn on n vertices are E – cordial if not congruent to 2 (mod 4).

3. Preliminaries:

1) Fusion of vertex. Let G be a (p,q) graph. Let xu and xv be two vertices of G. We replace them with single vertex w and all edges incident with u and that with v are made incident with w. If a loop is formed is deleted.

2) Path union of graph G i.e.Pm(G) is obtained by taking a path Pm and m copies of graph G. Fuse a copy each of G at every vertex of path at given fixed point on G. It has mp vertices and mq +m-1 edges, where G is a (p,q) graph. If we change the vertex on G that is fused with vertex of Pm then we generally get a path union non isomorphic to earlier structure. In this paper we define a e-cordial function f that does not depends on which vertex of given graph G is fused with vertex. This allows us to obtain path union in which the same graph G is fused with vertices of Pm at different vertices of G, as our choice and the same function f is applicable to all such structures that are possible on Pm(G).

3) Flag of a graph G denoted by FL(G) is obtained by taking a graph G=G(p,q). At suitable vertex of G attach a pendent edge. It has p+1 vertices and q+1 edges. We show below in diagram flag of K4, i.e.Fl(K4)

Fig 3.1 Fl(K4)

References

4) **Wheel graph W_{n}**. Take a cycle C_{n} and a vertex w not on C_{n}. Join every vertex of C_{n} by an edge each to w. The resultant graph is wheel W_{n}. Sometimes the same graph is shown by W_{n+1}. It has n+1 vertices and 2n edges. The edges incident with w are called as pokes and vertex w as hub. The other edges that lie on cycle C_{n} are called as cycle edges.

5) **Gear graph G_{n}**. It is obtained from wheel graph W_{n}. A new vertex is introduced on each cycle edge. Sometimes it is denoted by G_{n+1}. It has 2n+1 vertices and 3n edges.

![Fig 3.2 W_4](image)

![Fig 3.3 W_4](image)

### 4. Main Results Proved:

**Theorem 4.1**: \( G = P_{m}(W_{4}) \) is e-cordial iff \( m \) is not congruent to \( 2 \pmod{4} \).

**Proof**: We take a path \( P_{m} = (v_{1}, v_{2}, \ldots v_{m}) \) and at each of it’s vertex fuse a copy of \( W_{4} \) at a fixed vertex of \( W_{4} \). Define a function \( f : E(G) \rightarrow \{0, 1\} \) as follows: Label of every edge on path \( P_{m} \) is ‘0’. Using \( f \) we get following three labeled copies of \( W_{4} \): Type A, Type B and Type C. Of these type A only is e-cordial. We use them all suitably to obtain a labeled copy of \( G \).

![Fig 4.1](image)

![Fig 4.2](image)

![Fig 4.3](image)

On \( P_{m} \), for vertex \( v_{1} \), fuse type A label. This is e-cordial follows from fig 4.1. On \( v_{2} \) fuse type B label. At this stage label distribution is \( v_{2}(0,1) = (6,4) \), \( c_{0}(0,1) = (8,9) \). Note that it is not e-cordial.

At \( v_{3} \), fuse type C label. At this stage label distribution is \( v_{3}(0,1) = (7,8) \), \( c_{0}(0,1) = (13,13) \). At \( v_{4} \) use type B label. The label distribution is \( v_{4}(0,1) = (10,10) \), \( c_{0}(0,1) = (17,18) \). At \( v_{5} \) use type A label. Label distribution is \( v_{5}(0,1) = (13,12) \), \( c_{0}(0,1) = (22,22) \).

For \( i > 5 \) labeling is done in the following way.

**At** vertex \( v_{i} \) of path \( P_{m} \) fuse **type C** label if \( i \equiv 2 \pmod{4} \). Label distribution is \( v_{i}(0,1) = (14+10x, 16+10x) \), \( c_{0}(0,1) = (27+18x, 26+18x) \) for \( i = 4x+2, x = 2, 3, \ldots \)

**At** vertex \( v_{i} \) of path \( P_{m} \) fuse **type B** label if \( i \equiv 3 \pmod{4} \). Label distribution is \( v_{i}(0,1) = (17+10x, 18+10x) \), \( c_{0}(0,1) = (31+18x, 31+18x) \) for \( i = 4x+3, x = 2, 3, \ldots \)

**At** vertex \( v_{i} \) of path \( P_{m} \) fuse **type A** label if \( i \equiv 0 \pmod{4} \). Label distribution is \( v_{i}(0,1) = (20+10x, 20+10x) \), \( c_{0}(0,1) = (36+18x, 35+18x) \) for \( i = 4x, x = 2, 3, \ldots \)

At vertex \( v_{i} \) of path \( P_{m} \) fuse **type B** label if \( i \equiv 1 \pmod{4} \). Label distribution is \( v_{i}(0,1) = (23+10x, 22+10x) \), \( c_{0}(1) = (40+18x, 40+18x) \) for \( i = 4x+3, x = 2, 3, \ldots \)

That is here onwards sequence of type C, Type B, Type A, Type B is repeated. The function \( f \) is **not vertex sensitive** in the sense that which vertex on \( W_{4} \) is used to form path union is not important.

**Theorem 4.2**: \( G = P_{m}(G_{4}) \) is e-cordial iff \( m \) is not congruent to \( 2 \pmod{4} \).

**Proof**: We take a path \( P_{m} = (v_{1}, v_{2}, \ldots v_{m}) \) and at each of it’s vertex fuse a copy of \( G_{4} \) at a fixed vertex of \( G_{4} \). Define a function \( f : E(G) \rightarrow \{0, 1\} \) as follows: Label of every edge on path \( P_{m} \) is ‘0’. Using \( f \) we get following three labeled copies of \( G_{4} \): Type A, Type B and Type C. Of these type A only is e-cordial. We use them all suitably to obtain a labeled copy of \( G \).
On $P_m$, for vertex $v_1$ fuse type A label. On $v_2$ fuse type B label. The label number distribution for $P_3(G_4)$ is $\nu_i(0,1) = (8,10)$, $e_i(0,1) = (13,12)$.

At this stage $P_m(G)$ is not e-cordial. At vertex $v_3$ fuse type C label. The label number distribution for $P_3(G_4)$ is $\nu_i(0,1) = (13,14)$, $e_i(0,1) = (19,19)$. At vertex $v_4$ use type C label. Label distribution is $\nu_i(0,1) = (18,18)$, $e_i(0,1) = (25,26)$.

For all $i \geq 4$ and $x = 1, 2, \ldots$. We have:

- At vertex $v_i$ on $P_m$ fuse type A label if $i \equiv 1 \pmod{4}$. Label distribution is $\nu_i(0,1) = (23+18x, 22+18x)$, $e_i(0,1) = (32+26x, 32+26x)$; $i = 4x + 1$

- At vertex $v_i$ on $P_m$ fuse type B label if $i \equiv 2 \pmod{4}$. Label distribution is $\nu_i(0,1) = (26+18x, 28+18x)$, $e_i(0,1) = (39+26x, 38+26x)$. G is not e-cordial; $i = 4x + 2$

- At vertex $v_i$ on $P_m$ fuse type C label if $i \equiv 3 \pmod{4}$. Label distribution is $\nu_i(0,1) = (31+18x, 32+18x)$, $e_i(0,1) = (45+26x, 45+26x)$; $i = 4x + 3$

- At vertex $v_i$ on $P_m$ fuse type C label if $i \equiv 0 \pmod{4}$. Label distribution is $\nu_i(0,1) = (36+18x, 36+18x)$, $e_i(0,1) = (51+26x, 52+26x)$; $i = 4x + 4$. Thus the graph is not e-cordial if $m \equiv 2 \pmod{4}$. The function $f$ is not vertex sensitive in the sense that which vertex on $W_5$ is used to form path union is not important. (one may use hub vertex or cycle vertex to obtain path union). The same $f$ is applicable and gives us e-cordial labeling with same constraints.

**Theorem 4.3:** $G = P_m(W_5)$ is e-cordial iff $m$ is not congruent to 1(mod 2).

**Proof:** We take a path $P_m=\nu_1, \nu_2, \ldots \nu_m$ and at each of it’s vertex fuse a copy of $W_5$ at a fixed vertex of $W_5$. Define a function $f: E(G) \rightarrow \{0,1\}$ as follows: Label of every edge on path $P_m$ is ‘0’. Using $f$ we get following two labeled copies of $W_5$ Type A, Type B. We use them suitably to obtain a labeled copy of $G$.

At vertex $v_i$ fuse type A label if $i \equiv 1 \pmod{2}$. At vertex $v_i$ fuse type B label if $i \equiv 0 \pmod{2}$ The label number distribution for $P_{2x+1}(G_4)$ is $\nu_i(0,1) = (2+6x, 4+6x)$, $e_i(0,1) = (5+11x, 5+11x)$. Where $x = 2x+1$, $x = 0, 1, 2$. The label number distribution for $P_{2x}(G_4)$ is $\nu_i(0,1) = (6x, 6x)$, $e_i(0,1) = (10+11x, 11+11x)$. Where $x = 2x$, $x = 1, 2$. Thus the graph is e-cordial. The function $f$ is not vertex sensitive in the sense that which vertex on $W_5$ is used to form path union is not important (we can use hub vertex or cycle vertex to obtain path union). The same $f$ is applicable and gives us e-cordial labeling with same constraints.

**Theorem 4.4:** Let $G^{*}=f\{W_5\}$, $G = P_m(G^{*})$ is e-cordial iff $m$ is not congruent to 1(mod 2).

**Proof:** We take a path $P_m=\nu_1, \nu_2, \ldots \nu_m$ and at each of it’s vertex fuse a copy of $G^{*}$ at a fixed vertex of $G^{*}$. Define a function $f: E(G^{*}) \rightarrow \{0,1\}$ as follows: Label of every edge on path $P_m$ is ‘0’. Using $f$ we get following two labeled copies of $G^{*}$ Type A, Type B. Of these none is E- cordial. We use them suitably to obtain a labeled copy of $G$.
At vertex $v_i$ of $P_n$ use type A labeling if $i \equiv 1 \pmod{2}$ and Type B labeling otherwise. The label number distribution is $v_i(0,1) = (5+22x,6+22x)$, $e_i(0,1) = (7+32x,8+32x)$ when $m = 1+2x$, $x = 0, 1, 2, \ldots$ if $m$ is of type $2x$, $x = 0, 1, 2, \ldots$ if $m$ is of type $4$, $x = 0, 1, 2, \ldots$. If $m$ is of type $4x$, $x = 0, 1, 2, \ldots$. Then, $v_i(0,1) = (22x, 22x)$, $e_i(0,1) = (32x-1, 32x)$. If $m$ is of type $4x+2$, $x = 0, 2, \ldots$, $v_i(0,1) = (22x, 22x)$, $e_i(0,1) = (32x-1, 32x)$. If $m$ is of type $4x+3$, $x = 0, 2, \ldots$, $v_i(0,1) = (22x, 22x)$, $e_i(0,1) = (32x-1, 32x)$.

We Start with label A by fusing it with vertex $v_i$ of path $P_m$ for $i \equiv 1 \pmod{2}$ and Type B label for $i \equiv 0 \pmod{2}$. The resultant graph is e-cordial. The label number distribution is $v_i(0,1) = (6m, 6m)$ for all $m$ and $e_i(0,1) = (9m, 9m)$. Theorem 4.7: Let $G = m(0, 1)$ be a graph which is e-cordial if $m$ is not congruent to $0 \pmod{2}$.

Proof: We take a path $P_m = \{v_1, v_2, \ldots, v_m\}$ and at each of it’s vertex fuse a copy of $G_t$ at a fixed vertex of $G_s$. Define a function $f: E(G) \rightarrow \{0, 1\}$ as follows: Label of every edge on path $P_m$ is ‘0’. Using $f$ we get following two labeled copies of $G_t$ Type A, Type B. Of these none is e-cordial. We use them suitably to obtain a labeled copy of $G$. The label number distribution is $v_i(0,1) = (2+6m, 4+6m)$, $e_i(0,1) = (5+10m, 7+10m)$.
Theorem 4.8: Let $G = P_m(\text{Fl}(W_3))$ be e-cordial iff $m$ is not congruent to $2$ (mod 4).

Proof: We take a path $P_m = (v_1, v_2, \ldots, v_m)$ and at each of it’s vertex fuse a copy of $\text{Fl}(W_3)$ at a fixed vertex of $W_3$. Define a function $f: E(G) \rightarrow \{0, 1\}$ as follows: Using $f$ we get following labeled copy of $\text{Fl}(W_3)$: Type A, Type B. Of these Type A is $E$-cordial. We use them suitably to obtain a labeled copy of $G$.

Label of every edge on path $P_m$ is ‘0’. At vertex $v_i$ of $P_m$ fuse type A labeling if $i \equiv 1, 3, 4$ (mod 4) and Type B labeling if $i \equiv 2$ (mod 4). The label number distribution is $v_f(0,1) = (3+14x, 4+14x)$ for $m = 1+4x, x = 0, 1, 2, \ldots$ The label number distribution is $v_f(0,1) = (5+24x, 6+24x)$ for $m = 3+4x, x = 0, 1, 2, \ldots$ Further $e_f(0,1) = ((5+6(m-1), 6+6(m-1))$ for all $m = 1, 2, \ldots$

Conclusions:

In this paper we study path union of wheel related graphs and show that under certain conditions these graphs are e- cordial. We show that

1) $P_m(W_4)$ is e-cordial iff $m$ is not congruent to $2$ (mod 4).
2) $P_m(G_2)$ is e-cordial iff $m$ is not congruent to $2$ (mod 4).
3) $P_m(W_5)$ is e-cordial iff $m$ is not congruent to $1$ (mod 2).
4) $G = P_m(\text{fl}(W_3))$ is e-cordial iff $m$ is not congruent to $1$ (mod 2).
5) $G' = \text{fl}(G_2), G = P_m(G')$ is e-cordial iff $m$ is not congruent to $1$ (mod 2).
6) $G = P_m(G_2)$ is e-cordial iff $m$ is not congruent to $2$ (mod 4).
7) $G = P_m(\text{fl}(G_3))$ is e-cordial iff $m$ is not congruent to $2$ (mod 4).
8) $G = P_m(\text{fl}(W_3))$ is e-cordial iff $m$ is not congruent to $2$ (mod 4).

It is important to note that all path unions $P_m(G)$ of graph $G$ discussed above are not vertex sensitive in the sense that we can choose any vertex on $G$ to form $P_m(G)$ and the resultant graph is e-cordial. We don’t have to make any change in already defined function $f$.

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