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Clique Secure Domination in Graphs under Some Binary Operations[1]

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Abstract: Let G be a connected simple graph. A nonempty subset S of the vertex set $V(G)$ is a clique in G if the graph $\langle S\rangle$ induced by S is complete. A clique S in G is a clique dominating set if it is a dominating set. A clique dominating set S is a clique secure dominating set in G if for every vertex $u \in V(G) \setminus S$, there exists a vertex $v \in S \cap N_G(u)$, such that $(S \setminus \{v\}) \cup \{u\}$ is a dominating set in G. The clique secure domination number, denoted by $\gamma_{cls}(G)$, is the smallest cardinality of a clique secure dominating set in G. In this paper, we give the characterization of the clique secure dominating set resulting from the lexicographic and Cartesian products of two graphs and give some important results.

Mathematics Subject Classification: 065C69

Keywords: dominating set, clique dominating set, secure dominating set, clique secure dominating set

I. INTRODUCTION

Let G be a simple connected graph. A subset S of a vertex set $V(G)$ is a *dominating set* of G if for every vertex $v \in V(G)\setminus S$, there exists a vertex $x \in S$ such that xv is an edge of G. The *domination number* $\gamma(G)$ of G is the smallest cardinality of a dominating set S of G. Dominating sets have several applications in a variety of fields, including communication and electrical networks, protection and location strategies, data structures and others. For more background on dominating sets, the reader may refer to [1,2].

A complete graph of order n, denoted by K_n, is the graph in which every pair of its distinct vertices are joined by an edge. A nonempty subset S of $V(G)$ is a clique in G if the graph $\langle S\rangle$ induced by S is complete. A nonempty subset S of a vertex set $V(G)$ is a *clique dominating set* of G if S is a dominating set and S is a clique in G. The minimum cardinality among all clique dominating sets of G, denoted by $\gamma_{cl}(G)$, is called the clique domination number of G. A clique dominating set S of G with $|S| = \gamma_{cl}(G)$ is called a γ_{cl} *-set* of G. Clique dominating sets have a great diversity of applications. In setting up the communications links in a network one might want a strong core group that can communicate with each other member of the core group and so that everyone outside the core group could communicate with someone within the core group. A group of forest fire sentries that could see various sections of a forest might also be positioned in such a way that each could see the others in order to use triangulation to locate the site of a fire. In addition, the properties of dominating sets are useful in identifying structural properties of a social network [3,4]. Wolk [5] presents a forbidden subgraph characterization of a class of graphs which have a dominating clique of size one. He called such a dominating clique a central vertex or central point. The idea of Wolk was extended by Cozzens and Kelleher [6] to get forbidden subgraph conditions sufficient to imply the existence of a dominating set that induces a complete subgraph, a dominating clique. Daniel and Canoy [7] characterized the clique dominating sets in the join, corona, composition and Cartesian product of graphs and determine the corresponding clique domination number of the resulting graph. Other variants of clique domination in graphs are found in [8,9].

Another type of domination parameter is the secure domination in graphs. A dominating set S of $V(G)$ is a *secure dominating set* of G if for each $u \in V(G) \setminus S$, there exists $v \in S$ such that $uv \in E(G)$ and the set $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of G. The minimum cardinality of a secure dominating set of G, denoted by $\gamma_s(G)$, is called the *secure domination number* of G. Secure domination in graphs was studied and introduced by [10,11] and can be read in [12]. Secure dominating sets can be applied as protection strategies by minimizing the number of guards to secure a system so as to be cost effective as possible. . Other variants of secure domination in graphs can be read in [13,14,15,16,17,18].

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In [19], Kiunisala and Enriquez, introduced the clique secure domination in graphs. Accordingly, a clique dominating set S in a graph G is a *clique secure dominating set* of G if for every $u \in V(G) \setminus S$, there exists $v \in S \cap N_G(u)$ such that $(S \setminus \{v\}) \cup \{u\}$ is a dominating set in G. The *clique secure domination number* of G denoted by $\gamma_{ds}(G)$ is the minimum cardinality of a clique secure dominating set of G. A clique secure dominating set of cardinality $\gamma_{ds}(G)$ is called γ_{ds}*-set*. This motivate the researchers to extend the concept by introducing the clique secure dominating set resulting from the lexicographic and the Cartesian products of two graphs. For the general concepts, the readers may be referred to Chartrand and Zhang [20].

A *graph* G is a pair $(V(G), E(G))$, where $V(G)$ is a finite nonempty set called the *vertex-set* of G and $E(G)$ is a set of unordered pairs $\{u, v\}$ (or simply uv) of distinct elements from $V(G)$ called the *edge-set* of G. The elements of $V(G)$ are called *vertices* and the cardinality $|V(G)|$ of $V(G)$ is the *order* of G. The elements of $E(G)$ are called *edges* and the cardinality $|E(G)|$ of $E(G)$ is the *size* of G. If $|V(G)| = 1$, then G is called a trivial graph. If $E(G) = \emptyset$, then G is called an empty graph. The *open neighborhood* of a vertex $v \in V(G)$ is the set $N_G(v) = \{u \in V(G): uv \in E(G)\}$. The elements of $N_G(v)$ are called *neighbors* of v. The *closed neighborhood of* $v \in V(G)$ is the set $N_G[v] = N_G(v) \cup \{v\}$. If $X \subseteq V(G)$, the *open neighborhood* of X in G is the set $N_G(X) = \cup_{v \in X} N_G(v)$. The *closed neighborhood* of X in G is the set $N_G[X] = \cup_{v \in X} N_G[v] = N_G(X) \cup X$. When no confusion arises, $N_G[x]$ [resp. $N_G(x)$] will be denoted by $N[x]$ [resp. $N(x)$]. Let $x, y \in V(G)$. Any x-y path of length equal to $d_G(x, y)$ (the distance between the vertices x and y in G) is called an x-y *geodesic*. The interval $I[x, y] = I_G[x, y]$ consists of x, y and all vertices lying on any x-y geodesic. If $S \subseteq V(G)$, then the *geodetic closure* of S is the set $I[S] = I_G[S] = \cup_{x,y \in S} I[x, y]$. S is *convex* if $I[x, y] \subseteq S$ for any $x, y \in S$, i.e., $I_G[S] = S$. Unless otherwise stated, all graphs in this paper are assumed to be simple and connected.

II. RESULTS

In this paper, we denote by $\mathcal{CS}(G)$, a family of all graph G with clique secure dominating set. Thus, for the purpose of this study, it is assumed that all connected nontrivial graphs considered belong to the family $\mathcal{CS}(G)$.

Remark 2.1 *A clique secure dominating set S of a graph G is a clique and a secure dominating set of G.*

A clique dominating set S in a graph G is a *secure clique dominating set* of G if for every $u \in V(G) \setminus S$, there exists $v \in S \cap N_G(u)$ such that $(S \setminus \{v\}) \cup \{u\}$ is a clique dominating set in G.

Lemma 2.2 [19] *Every secure clique dominating set of a graph G is a clique secure dominating set of G.*

The converse of Remark 2 is not true. Consider the graph in Figure 1.

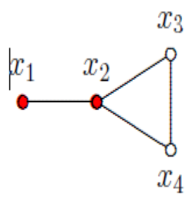

Figure 1: A graph G with $\gamma_{cls}(G) = 2$

The set $S = \{x_1, x_2\}$ is a clique secure dominating set but not a secure clique dominating set of a graph G. In fact G has no secure clique dominating set.

Remark 2.3 *Let G be a non-trivial connected graph. Then $\gamma(G) \leq \gamma_d(G) \leq \gamma_{ds}(G)$.*

The *lexicographic product* of two graphs G and H is the graph $G[H]$ with vertex-set $V(G[H]) = V(G) \times V(H)$ and edge-set $E(G[H])$ satisfying the following conditions: $(x, u)(y, v) \in E(G[H])$ if and only if either $xy \in E(G)$ or $x = y$ and $uv \in E(H)$.

A subset C of $V(G[H]) = V(G) \times V(H)$ can be written as $C = \cup_{x \in S} [\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for every $x \in S$. We shall be using this form to denote any subset C of $V(G[H])$.

Remark 2.4 *If G and H are complete graphs, then $\gamma_{ds}(G[H]) = 1$.*

The following result is needed for the characterization of the clique secure dominating sets of the composition of two graphs.

Theorem 2.5 (Daniel and Canoy 2015) *Let G and H be connected nontrivial graphs such that G has a clique dominating set. A subset $C = \cup_{x \in S} [\{x\} \times T_x]$ where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a clique dominating set of $G[H]$ if and only if S is a clique dominating set of G such that*

(i) $\langle T_x \rangle$ is a clique in H for each $x \in S$ and
(ii) T_x is a dominating set of H whenever $S = \{x\}$.

The following result is the characterization of the clique secure dominating sets of the composition of two graphs.

Theorem 2.6 *Let G and H be connected non-complete graphs such that G has a clique dominating set. A subset $C = \cup_{x \in S} [\{x\} \times T_x]$ where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a clique secure dominating set of $G[H]$ if and only if one of the following statements is satisfied:*

(i) S is a secure clique dominating set of G and $|T_x| = 1$ for all $x \in S$ with $diam(H) \leq 4$.
(ii) S is a clique secure dominating set of G and $\langle T_x \rangle$ is a clique in H with $|T_x| \geq 2$ for all $x \in S$ where
T_x is a clique secure dominating set of H whenever $S = \{x\}$ is a dominating set of G.

Proof. Suppose that a subset $C = \cup_{x \in S} [\{x\} \times T_x]$ where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a clique secure dominating set of $G[H]$. Then C is a clique and secure dominating set of $G[H]$ by Remark 2.1. Thus, S is a clique dominating set of G such that $\langle T_x \rangle$ is a clique in H for each $x \in S$ and T_x is a dominating set of H whenever $S = \{x\}$ by Theorem 2.5.

Let $T_x = \{a\}$ and $|S| \geq 2$. Suppose that S is not a secure clique dominating set of G. Let $u \in V(G) \setminus S$ such that $uv \in E(G)$ for some $v \in S$. Since S is a clique dominating set of G, it follows that $S_u = (S \setminus \{v\}) \cup \{u\}$ is not a clique dominating set of G. If S_u is not dominating, then $S_u \times \{a\}$ is not a dominating set of $G[H]$. This implies that $C^* = (C \setminus \{v, a\}) \cup \{(u, a)\} = [(S \setminus \{v\}) \cup \{u\}] \times \{a\} = S_u \times \{a\}$ is not a dominating set in $G[H]$ contrary to our assumption that C is a clique secure dominating set in $G[H]$. If S_u is a dominating set of G then S_u is not a clique set in G. This implies that $S_u \times \{a\}$ is not a dominating set for some $a \in V(H)$. In particular, if $T_x = \{a\}$ is not a dominating set of H, then $C^* = (C \setminus \{v, a\}) \cup \{(u, a)\} = [(S \setminus \{v\}) \cup \{u\}] \times \{a\} = S_u \times \{a\}$ is not a dominating set in $G[H]$ contrary to our assumption that C is a clique secure dominating set in $G[H]$. Thus, S must be a secure clique dominating set of G.

Now, let $S = \{x, y\}$ and $T_v = \{a\}$ for all $v \in S$. Suppose that the $diam(H) \geq 5$, say $diam(H) = 5$. Then there exists $e \in V(H)$ such that $d_H(a, e) = 5$. Since S is a clique dominating set of G, it follows that $C = \cup_{v \in S} [\{v\} \times T_v] = \{(x, a), (y, a)\}$ is a clique dominating set of $G[H]$ by Theorem 2.5. This implies that $C^* = (C \setminus$ □,□□ □,□}={(□,□),(□,□)}. Since □□[□]((□,□),(□,□))=5 there exists □□ □□□{□,□}) such that $(y, a)(y, c), (y, c)(y, e) \notin (E(G[H]))$. This implies that C^* is not a dominating set contrary to our assumption that C is a clique secure dominating set of $G[H]$. Similarly, if $diam(H) > 5$, then C^* is not a dominating set contrary to our assumption. Thus, $diam(H) \leq 4$ and this proves condition (i).

Assume that S is not a secure clique dominating set of G. Let $x \in S$ and $z \in V(G) \setminus S$ such that $xz \in E(G)$ and $zy \notin E(G)$ for all $y \in S \setminus \{x\}$. Suppose that $T_x = \{b\}$ for all $x \in S$. Since H is non-complete, let $a, c \in N_H(b)$ such that $ac \notin E(H)$. Then $C^* = (C \setminus \{(x, b)\}) \cup \{(z, a)\}$. Since $(z, a)(z, c), (y, b)(z, c) \notin E(G[H])$ for all $y \in S \setminus \{x\}$, it follows that (z, c) is not adjacent to any element of C^*. This implies that C^* is not a dominating set of $G[H]$ contrary to our assumption that C is a clique secure dominating set of $G[H]$. Thus, $|T_x| \neq 1$ for all $x \in S$. Since C is a clique secure dominating set, $C^* = (C \setminus \{(x, b)\}) \cup \{(z, a)\} =$

$[(S \setminus \{x\}) \cup \{z\}] \times T_v$ is a dominating set of $G[H]$ for all $v \in S \setminus \{x\}$. This implies that $(S \setminus \{x\}) \cup \{z\}$ is a dominating set of G. Since S is a clique dominating set, it follows that S is a clique secure dominating set of G.

Now, let $S = \{x\}$. Suppose that $T_x = \{a\}$. Since S and T_x are dominating sets in G and H respectively, it follows that $C = \{(x,a)\}$ is a dominating set of $G[H]$. Since G is non-complete, let $y, z \in N_G(x)$ such that $yz \notin E(G)$. Then $(y,a)(z,a) \notin E(G[H])$. Thus, $C^* = \{(y,a)\}$ is not a dominating set of $G[H]$ contrary to our assumption that C is a clique secure dominating set of $G[H]$. Thus $|T_x| \neq 1$ and hence $|T_x| \geq 2$. If T_x is not a clique set in H, then there exists $a, b \in T_x$ such that $ab \notin E(H)$. This implies that $(x,a)(x,b) \notin E(G[H])$. Thus C is not a clique secure dominating set of $G[H]$ contrary to our assumption. Therefore, T_x must be a clique set in H and hence T_x is a clique dominating set of H. Let $c \in T_x \setminus \{a\}$. Suppose that T_x is not a secure dominating set of H. Let $T_x = \langle a \rangle + \underline{K}_3$. Then $(T_x \setminus \{a\}) \cup \{c\}$ is not a dominating set of H. Thus, $C^* = (C \setminus \{(x,a)\}) \cup \{(x,c)\} = \{x\} \times [(T_x \setminus \{a\}) \cup \{c\}]$ is not a dominating set of $G[H]$ contrary to our assumption that C is a clique secure dominating set of $G[H]$. This implies that T_x must be a secure dominating set of H, and hence T_x is a clique secure dominating set of H. This complete the proofs of statement (ii).

For the converse, suppose that the statement (i) or (ii) is satisfied. Then a subset $C = \cup_{x \in S} [\{x\} \times T_x]$ where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a clique dominating set of $G[H]$ by Theorem 2.5.

Suppose first that statement (i) holds. Let $z \in V(G) \setminus S$. Since S is a secure clique dominating set of G, $S_z = (S \setminus \{x\}) \cup \{z\}$ is a clique dominating set of G for some $x \in S$. Let $T_x = \{a\}$. Then $C^* = S_z \times T_x = [(S \setminus \{x\}) \cup \{z\}] \times \{a\} = (C \setminus \{(x,a)\}) \cup \{(z,a)\}$ is a clique dominating set of $G[H]$. Since C is a clique dominating set of $G[H]$, it follows that C is a secure clique dominating set and hence a clique secure dominating set of $G[H]$ by Remark 2.2. Further, since *diam* $(H) \leq 4$, there exists $b, c \in V(H)$ such that $d_H(b,c) = 4$. If $b = a$ and $x \in S$, then $C^* = (C \setminus \{(x,a)\}) \cup \{(y,c)\}$ for some $y \in S$ with $xy \in E(G)$. Since $d_H(b,c) = 4$, it follows that $d_{G[H]}((y,a),(y,c)) = 4$. Thus, $N_{G[H]}[C^*] = N_{G[H]}[(C \setminus \{(x,a)\}) \cup \{(y,c)\}] = N_{G[H]}[(C \setminus$ ▯,▯▯ ▯▯[▯]][{(▯,▯)}] = ▯▯[▯]▯▯▯× ▯▯ ▯▯[▯]][{(▯}× {▯}]. Since ▯ is a secure clique dominating set, ▯▯{▯} is a dominating set of G. This implies that

$$N_{G[H]}[(S \setminus \{x\}) \times \{a\}] = [(S \setminus \{x\}) \times \{a\}] \cup (\cup_{v \in N_G(S \setminus \{x\})} [\{v\} \times V(H)]) \cup [\{y\} \times N_H(a)].$$
and $$N_{G[H]}[\{y\} \times \{c\}] = [\{y\} \times \{c\}] \cup (\cup_{v \in N_G(\{y\})} [\{v\} \times V(H)]) \cup (\{y\} \times N_H(c)).$$
$$N_{G[H]}[(S \setminus \{x\}) \times \{a\}] = [(S \setminus \{x\}) \times \{a\}] \cup (\cup_{v \in N_G(S \setminus \{x\})} [\{v\} \times V(H)]) \cup [\{y\} \times N_H(a)].$$
and $$N_{G[H]}[\{y\} \times \{c\}] = [\{y\} \times \{c\}] \cup (\cup_{v \in N_G(\{y\})} [\{v\} \times V(H)]) \cup (\{y\} \times N_H(c)).$$

Thus, by routine computations, $N_{G[H]}[(S \setminus \{(x)\}) \times \{a\}] \cup N_{G[H]}[\{(y\} \times \{c\}] = V(G[H])$ and hence $N_{G[H]}[C^*] = V(G[H])$. This implies that C^* is a dominating set of $G[H]$, and hence, C is a secure dominating set of $G[H]$. Since C is also a clique dominating set, it follows that C is a clique secure dominating set of $G[H]$ by Remark 2.1.

Next, suppose that statement (ii) holds. Consider that $S = \{x\}$ is a dominating set of G and T_x is a clique secure dominating set of H. Then $C = \cup_{x \in S} [\{x\} \times T_x] = \{x\} \times T_x$. Let $a \in T_x$ and $c \in V(H) \setminus T_x$ such that $ac \in E(H)$. Then $(y,c) \in V(G[H]) \setminus C$ for some $y \in V(G)$. If $y = x$, then $(x,a)(x,c) \in E(G[H])$ and $C^* = (C \setminus \{(x,a)\}) \cup \{(x,c)\} = \{x\} \times [(T_x \setminus \{a\}) \cup \{c\}]$. Since T_x is a clique secure dominating set of H, it follows that $(T_x \setminus \{a\}) \cup \{c\}$ is a dominating set of H. This implies that C^* is a dominating set of $G[H]$. Since C is a clique dominating set of $G[H]$, it follows that C is a clique secure dominating set of $G[H]$. Now, suppose that $y \neq x$. Since $\{x\}$ is dominating, $xy \in E(G)$ for all $y \in V(G)$. Thus, $(x,a)(y,c) \in E(G[H])$ and $C^* = (C \setminus \{(x,a)\}) \cup \{(y,c)\} = (\{x\} \times [T_x \setminus \{a\}]) \cup \{(y,c)\}$. Thus, $N_{G[H]}[C^*] = N_{G[H]}[\{x\} \times (T_x \setminus \{a\})] \cup N_{G[H]}[(y,c)]$.
Since $N_{G[H]}[\{x\} \times (T_x \setminus \{a\})] = \{y\} \times V(H)$ *for all* $y \in V(G) \setminus \{x\}$ *and* $N_{G[H]}[(y,c)] = [\{x\} \times$ ▯▯▯ [▯▯▯ ▯▯(▯)(▯,▯)], it follows that ▯▯[▯][▯▯]=▯(▯[▯]) by routine computations and hence ▯▯ is a dominating set of $G[H]$. Since C is a clique dominating set of $G[H]$, it follows that C is a clique secure dominating set of $G[H]$.

Now, consider $|S| \geq 2$ and S is a clique secure dominating set of G and $\langle T_x \rangle$ is a clique in H with $|T_x| \geq 2$ for all $x \in S$. Let $a \in T_x$ and $c \in V(H) \setminus T_x$ for all $x \in S$. If $ac \notin E(H)$, then let $u \in V(G) \setminus S$ such that

$uv \in E(G)$ for some $v \in S$. Then for all $(u,c) \in V(G[H]) \setminus C$, $(v,a)(u,c) \in E(G[H])$ for some $(v,a) \in C$. Thus, $C^* = (C \setminus \{v,a\}) \cup \{(u,c)\} = [(S \setminus \{v\}) \times \{a\}] \cup [S \times (T_x \setminus \{a\})] \cup \{(u,c)\}$ *for all* $x \in S$.

Since $|S| \geq 2$ and S is a dominating set of G, it follows that $S \times (T_x \setminus \{a\})$ is a dominating set of $G[H]$. Thus, C^* is a dominating set of $G[H]$. Since C is a clique dominating set of $G[H]$, it follows that C is a clique secure dominating set of $G[H]$. Finally, suppose that $ac \in E(H)$. Then $(v,c) \in V(G[H]) \setminus C$ and $(v,a)(v,c) \in E(G[H])$ for some $(v,a) \in C$. Thus, $C^* = (C \setminus \{v,a\}) \cup \{(v,c)\} = [S \times (T_x \setminus \{a\})] \cup [(S \setminus \{v\}) \times$ □□□,□ □□□ □□□ □□ □. By similar arguments used above, □ is a clique secure dominating set of □[□]. This complete the proofs. ■

Corollary 2.7 *Let G and H be connected non-complete graphs with* $\gamma(G) = 1$. *Then*

$$\gamma_{ds}(G[H]) = \{2 \; if \; \gamma(H) \; \gamma_{cl}(H) \; if \; if \; H \; has \; a \; clique \; dominating \; set$$

Proof Let $S = \{x\}$ be a γ-*set* of G. Then G has a clique dominating set. Suppose that $\gamma(H) = 1$. Let T_x be a clique dominating set of H. If $T_x = \{a\}$, then $C = \{(x,a)\}$ is not a secure dominating set of $G[H]$ by the proof of Theorem 2.6. Thus $|T_x| \geq 2$. This implies that a subset $C = \cup_{x \in S} [\{x\} \times T_x]$ is a clique secure dominating set of $G[H]$ by Theorem 2.6. Thus, $\gamma_{ds}(G[H]) \leq |C|$. Further, $|C| = |S||T_x| \geq 2$. *If* $|T_x| = 2$, then $|C| = 2$, that is, $\gamma_{ds}(G[H]) \leq 2$. Since G and H are non-complete graphs, $\gamma_{ds}(G[H]) \geq 2$ by Remark 2.4. Thus, $\gamma_{ds}(G[H]) = 2$. Suppose that $\gamma(H) \neq 1$ and H has a clique dominating set. Let T_x be a γ_{cl} *-set* of H. Then $\gamma_{cl}(H) = |T_x| \geq 2$. Thus, $C = \cup_{x \in S} [\{x\} \times T_x]$ is a clique secure dominating set of $G[H]$ by Theorem 2.6, that is, $\gamma_{ds}(G[H]) \leq |C|$. Since $|C| = |S||T_x| = \gamma_{cl}(H)$, it follows that $\gamma_{ds}(G[H]) \leq \gamma_{cl}(H)$. Since $\gamma_{ds}(G[H]) = \gamma_{ds}(G)\gamma_{ds}(H) \geq \gamma_{cl}(G)\gamma_{cl}(H) = 1 \cdot \gamma_{cl}(H) = \gamma_{cl}(H)$, it follows that $\gamma_{ds}(G[H]) = \gamma_{cl}(H)$. ■

The Cartesian product $G \boxdot H$ of two graphs G and H is the graph with $V(G \boxdot H) = V(G) \times V(H)$ and $(u,u')(v,v') \in E(G \boxdot H)$ if and only if either $uv \in E(G)$ and $u' = v'$ or $u = v$ and $u'v' \in E(H)$. Note that if $C \subseteq V(G \times H)$, then the G-*projection* and H-*projection* of C are, respectively, the sets

$$C_G = \{u \in V(G) : (u,b) \in C \; for \; some \; b \in V(H)\} \text{ and } C_H = \{v \in V(H) : (a,v) \in C \; for \; some \; a \in V(G)\}.$$

We need the following Theorem for our next characterization.

Theorem 2.8 (Daniel and Canoy 2015) *Let G and H be connected nontrivial graphs of orders* m *and* n, *respectively. Then* $G \boxdot H$ *has a clique dominating set if and only if either G is complete and* $\gamma(H) = 1$ *or H is complete and* $\gamma(G) = 1$. *Moreover,*

$$\gamma_{cl}(G \boxdot H) \{m \; if \; is \; complete \; and \; \gamma(H) = 1 \; n \; if \; is \; complete \; and \; \gamma(G) = 1 \; min \{m,n\} \; if \; G \; and \; H \; are \; both \; complete$$

The following result is the characterization of the clique secure dominating sets of the composition of two graphs.

Theorem 2.9 *Let G and H be connected nontrivial graphs. The* $G \boxdot H$ *has a clique secure dominating set if and only if G and H are complete graphs.*

Proof . Suppose that $G \boxdot H$ has a clique secure dominating set. Then $G \boxdot H$ has a clique dominating set. Thus, either G is complete and $\gamma(H) = 1$ or H is complete and $\gamma(G) = 1$ by Theorem 2.8. Thus, If G and H are complete graphs holds.

For the converse, suppose that G and H are complete graphs. Then $G \boxdot H$ has a clique dominating set by Theorem 2.8. Let $C = V(G) \times \{a\}$ for all $a \in V(H)$ be a clique dominating set of $G \boxdot H$. If $b \in V(H) \setminus \{a\}$, then $ab \in E(H)$ since H is complete, that is, $(x,a)(x,b) \in E(G \boxdot H])$ for all $x \in V(G)$. Thus,

$$C^* = (C \setminus \{(x,a)\}) \cup \{(x,b)\} = [(V(G) \setminus \{x\}) \times \{a\}] \cup \{(x,b)\}$$

$$N_{G\boxdot H}[C^*] = N_{G\boxdot H}[(V(G) \setminus \{x\}) \times \{a\}] \cup N_{G\boxdot H}[\{(x,b)\}]$$

$$= [(V(G) \setminus \{x\}) \times V(H)] \cup N_{G\boxdot H}[\{x\} \times V(H)] = V(G) \times V(H) = V(G \boxdot H).$$

This implies that C^* is a dominating set of $G \boxdot H$ and hence C is a clique secure dominating set of $G \boxdot H$. Similarly, if $C = \{x\} \times V(H)$ for all $x \in V(G)$ is a clique dominating set of $G \boxdot H$, then C is a clique secure dominating set of $G \boxdot H$. ■

The following result is an immediate consequence of Theorem 2.9.

Corollary 2.10 *Let G and H be connected nontrivial graphs. If G and H are complete graphs, then* $G \boxdot H = min\ \{|V(G)|, |V(H)|\}$

Proof . Suppose that G and H are complete graphs. Then the $G \boxdot H$ has a clique secure dominating set by Theorem 2.9. Let $C_1 = V(G) \times \{a\}$ for all $a \in V(H)$ and $C_2 = \{x\} \times V(H)$ for all $x \in V(G)$ be clique secure dominating sets of $G[H]$. Then $\gamma_{ds}\ (G \boxdot H) \leq |C_1|$ and $\gamma_{ds}\ (G \boxdot H) \leq |C_1|$. Thus,

$$\gamma_{ds}\ (G \boxdot H) \leq min\ \{|C_1|, |C_2|\} = min\ \{|V(G)|, |V(H)|\}.$$

Further, $\gamma_{ds}\ (G \boxdot H) \geq \gamma_d\ (G \boxdot H) = min\ \{|V(G)|, |V(H)|\}$ by Remark 2.3 and Theorem 2.8. Therefore, $\gamma_{ds}\ (G\ H) = min\ \{|V(G)|, |V(H)|\}$. ■

I.

III. CONCLUSION

A clique secure dominating set is a new variant of domination in graphs which was introduced by Kiunisala and Enriquez. Since this is new, further investigations must be done to come up with some substantial results on this domination in graphs. Thus, the clique secure dominating set resulting from the lexicographic and the Cartesian products of two graphs is the main purpose of this study. Moreover, domination in graphs is rich with immediate applications in the real world such as routing problems in internets, problems in electrical networks, data structures, neural and communication networks, protection and location strategies and many others. The clique secure domination in graph is not far from these applications. Hence, we looked forward for some real world application on the clique secure domination in graphs.

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