International Journal of Latest Engineering Research and Applications (IJLERA) ISSN: 2455-7137

Volume – 03, Issue – 07, July 2018, PP – 25-29

Perfect Outer-convex Domination in Graphs

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Abstract: Let G be a connected simple graph. A dominating set $S \subseteq V(G)$ is called a perfect dominating set of G if each $u \in V(G) \setminus S$ is dominated by exactly one element of S. A perfect dominating set S of a graph G is a perfect outer-convex dominating set if $V(G) \setminus S$ is convex. In this paper, we give the characterization of the perfect outer-convex domination number 1. Further, we give the characterization of the perfect outer-convex dominating set in the corona of two graphs and give its corresponding perfect outer-convex domination number.

Keywords: dominating set, convex dominating set, perfect dominating set, perfect outer-connected dominating set, perfect outer- convex dominating set

I. INTRODUCTION

The theory of domination is an area in graph theory with numerous research endeavors. A study on games and recreational activities contributes in making further study about domination in graph theory. In 1998, Haynes, Hedetniemi and Slater published a book which lists 1,222 papers in this area [1]. Varieties of domination parameters make an essential part to the researcher's motivation in doing research to this field. There are more than 75 variations of domination cited in the book [1]. These variations are constructed by creating additional conditions on S,$V(G)\setminus S$ or $V(G)$. Aside from varieties of domination parameters, scholars are also motivated to conduct researches in this field due to its wide and significant applications. Domination theory utilizations are seen and appreciated in diverse fields which include electrical engineering, operations research, biology, biochemistry, social sciences, education, computer science and in many other areas. In this paper, we develop a new variant of domination called perfect outer-convex dominating set and characterize it in the corona of two graphs.

Let G be a simple connected graph. A set of vertices S is a dominating set of G if every vertex of G is dominated by at least one vertex of S. The domination number $\gamma(G)$of G is the smallest cardinality of a dominating set S of G. Domination problem plays a vital role in the area of graph theory and combinatorial optimization. It has various applications in location problems, social network theory, computer communication networks and many others [2]. Domination in graph was introduced by Claude Berge in 1958 and Oystein Ore in 1962 [3].

A graph G is connected if there is at least one path that connects every two vertices $x, y \in V(G)$, otherwise, G is disconnected. For any two vertices u and v in a connected graph, the distance $d_G(u,v)$ between u and v is the length of a shortest path in G. A $u-v$ path of length $d_G(u,v)$ is also referred to as $u-v$ geodesic. The closed interval $I_G[u,v]$ consist of all those vertices lying on a $u-v$ geodesic in G. For a subset S of vertices of G, the union of all sets $I_G[u,v]$ for $u,v \in S$ is denoted by $I_G[S]$. Hence $x \in I_G[S]$ if and only if x lies on some $u-v$ geodesic, where $u,v \in S$. A set S is convex if $I_G[S] = S$. Certainly, if G is connected graph, then $V(G)$ is convex. Convexity in graphs was studied in [4, 5, 6].

A dominating set S which is also convex is called a convex dominating set of G. The convex domination number $\gamma_{con}(G)$ of G is the smallest cardinality of a convex dominating set of G. A convex dominating set of cardinality $\gamma_{con}(G)$ is called a $\gamma_{con}-set$ of G. Convex domination in graphs has been studied in [7, 8].

A dominating set $S \subseteq V(G)$ is called a perfect dominating set of G if each $u \in V(G) \setminus S$ is dominated by exactly one element of S. The perfect domination number of G, denoted by $\gamma_p(G)$, is the minimum cardinality of a perfect dominating set of G. A perfect dominating set of cardinality $\gamma_p(G)$ is called a γ_p-set of G. The perfect domination in graphs was introduced by Cockayne et.al [9]. A perfect dominating set S is a perfect outer-connected dominating set if $\langle V(G)\setminus S\rangle$ is connected. The concept of perfect outer-connected domination in graphs was introduced by [10].

Motivated by the definition of convex domination, perfect domination and perfect outer-connected domination in graphs, we define a new domination parameter in graphs called perfect outer-convex domination. A perfect dominating set S of a graph G is a *perfect* outer-convex dominating set if $V(G) \setminus S$ is convex. The perfect outer-convex domination number of G, denoted by $\tilde{\gamma}_{con}^{p}(G)$, is the smallest cardinality of a perfect outer-convex dominating set S of G. A perfect outer-convex dominating set with cardinality $\tilde{\gamma}_{con}^{p}(G)$ is called $\tilde{\gamma}_{con}^{p}$-set of G.

For general concepts we refer the reader to [11].

II. RESULTS

From the definitions, the following result is immediate.

Remark 2.1 Let G be a connected nontrivial graph of order $n \geq 2$. Then

i. $\gamma(G) \leq \gamma_p(G) \leq \tilde{\gamma}_{con}^{p}(G)$

ii. $1 \leq \tilde{\gamma}_{con}^{p}(G) \leq n-1$

Theorem 2.2 Let a, b, c, and n ≥ 2 be positive integers such that $a \leq b \leq c \leq n-1$. Then there exists a connected graph G with $|V(G)| = n$ such that $\gamma(G) = a, \gamma_p(G) = b$, and $\tilde{\gamma}_{con}^{p}(G) = c$.

Proof. Consider the following cases:

Case 1. Suppose a = b = c < n − 1.

Let G = Pa ∘ K_3 (see Fig. 1) and let $n = 4a$.

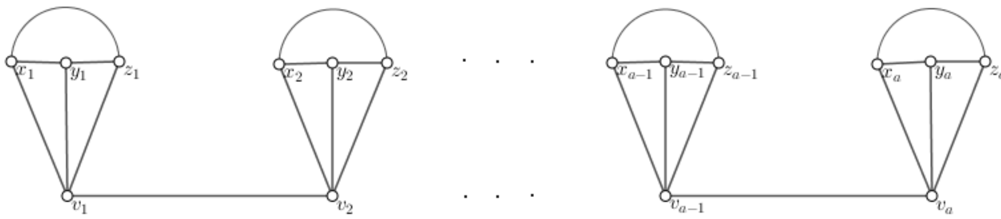

Figure 1: A graph G with a = b = c < n – 1

The set $A = \{x_i : i = 1, 2, \ldots, a\}$ is a γ -set, γ_p-set, and $\tilde{\gamma}_{con}^{p}$ -set of G. Thus, $|V(G)| = 4|A| = 4a = n$, $\gamma(G) = |A| = a, \gamma_p(G) = |A| = b$, and $\tilde{\gamma}_{con}^{p}$ (G) = |A| = c.

Case 2. Suppose a = b < c < n – 1

Let $c = 2a$ and $n = 5a$. Consider the graph G obtained from the graph in Figure 1 by adding the vertex r_i such that $r_i y_i \in E(G)$ where $i = 1,2,\ldots,a$ (see Fig. 2).

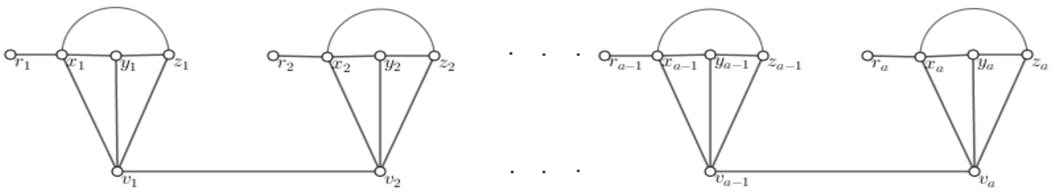

Figure 2: A graph G with $a = b < c < n - 1$

The set $A = \{x_i : i = 1,2,\ldots,a\}$ is $\gamma - set$ and $\gamma_p - set$, and $B = A \cup \{r_i : i = 1,2,\ldots,a\}$ is a $\tilde{\gamma}_{con}^{p} - set$. Thus, $|V(G)| = 5|A| = 5a = n$, $\gamma(G) = |A| = a$, $\gamma_p(G) = |A| = a = b$, and $\tilde{\gamma}_{con}^{p}(G) = |B| = 2a = c$.

Case 3: Suppose $a < b = c < n - 1$.

Let $a = 2k + 1$ for some $k \in \mathbb{N}$, $2b = 3a - 1, and\ n = 3a - 1$. Consider the graph G obtained from $P_a = [v_1, v_2, \ldots, v_a]$ and $P_{2a-1} = [x_1, y_2, \ldots, x_{a-1}, y_{a-1}\ x_a]$ by adding the edges $v_i x_i$ and $y_{a-2} y_{a-1}$ for all $i = 1,2,\ldots,a$ (see Fig. 3).

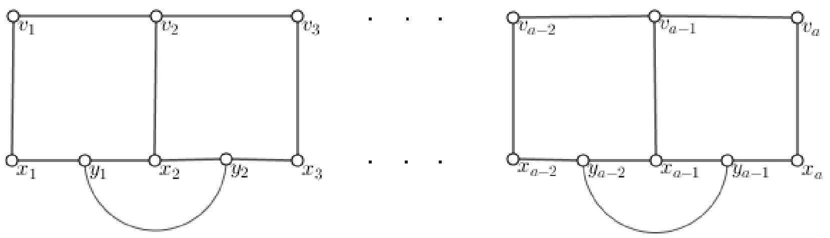

Figure 3: A graph G with $a < b = c < n - 1$

The set $A = \{x_i : i = 1,2,\dots,a\}$ is a $\gamma - set$ of G, $B = V(P_a) \cup \{x_{2i} : i = 1,2,\dots,\frac{a-1}{2}\}$ is a $\gamma_p - set$. Thus, $|V(G)| = |V(P_a)| + |V(P_{2a-1})| = a + 2a - 1 = 3a - 1 = n, \gamma(G) = |A| = a, \gamma_p(G) = |B| = a + \frac{a-1}{2} = \frac{3a-1}{2} = b$, and $\tilde{\gamma}^p_{con}(G) = |B| = b = c$.

Case 4: Suppose $a < b < c < n - 1$.

Let $a = 2k + 1$ for some $k \in \mathbb{N}, 2b = 3a - 1$, and $2c = 2n - 3a + 1$. Consider the graph G obtained from the graph in Fig. 3 by adding the vertex z_j such that $v_a z_j \in E(G)$ for all $j = 1,2,\dots,n - 3a + 1$ (see Fig. 4).

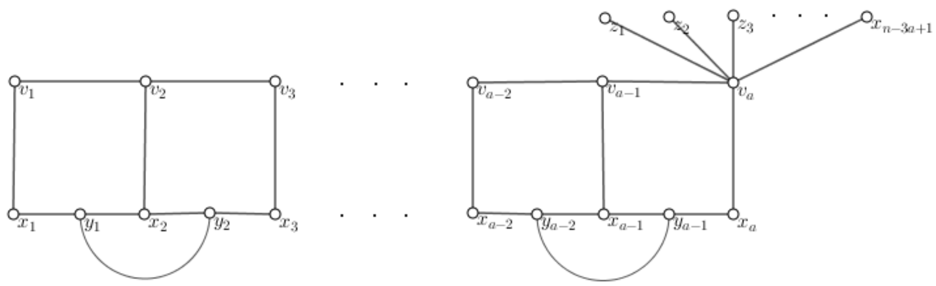

Figure 4: A graph G with $a < b < c < n - 1$

The set $A = \{x_i : i = 1,2,3,\dots,a-1\} \cup \{v_a\}$ is $\gamma - set$ of G, $B = V(P_a) \cup \left\{x_{2i} : i = 1,2,\dots,\frac{a-1}{2}\right\}$ is a $\gamma_p - set$ and $C = B \cup \{z_j : j = 1,2,\dots,n - 3a + 1\}$ is a $\tilde{\gamma}^p_{con} - set$. Thus $|V(G)| = |V(P_a)| + |V(P_{2a-1})| + \left|\{z_j : j = 1,2,\dots,n - 3a + 1\}\right| = (a) + (2a - 1) + (n - 3a + 1) = n, \gamma(G) = |A| = (a - 1) + 1 = a, \gamma_p(G) = |B| = (a) + (\frac{a-1}{2}) = \frac{3a-1}{2} = b$, and $\tilde{\gamma}^p_{con}(G) = |C| = b + (n - 3a + 1) = \frac{3a-1}{2} + n - 3a + 1 = \frac{2n-3a+1}{2} = c$.

Case 5: Suppose $a < b < c = n - 1$.

Let $a = 2k + 1$ for some $k \in \mathbb{N}, 2b = 3a - 1$, and $n = c + 1$. Consider the graph G obtained from Fig. 3 by adding the vertex z_j such that $v_a z_j \in E(G)$ for all $j = 1,2,\dots,c - \frac{7a-3}{2} + 1$ and adding the vertex u_k such that $u_k x_i \in E(G)$where $i = 1,2,\dots,a$ and for all $k = 1,2,\dots,\frac{a+1}{2}$ (see Fig. 5).

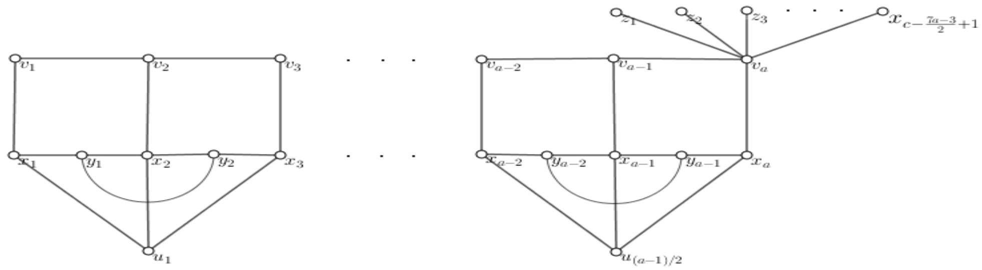

Figure 5: A graph G with $a < b < c = n - 1$

The set $A = \{x_{2i-1} : i = 1,2,3,\dots,a-1\} \cup \{v_a\}$ is a $\gamma - set$ of G, $B = V(P_a) \cup \{x_{2i} : i = 1,2,\dots,\frac{a-1}{2}\}$ is a $\gamma_p - set$ and $C = V(P_a) \cup V(P_{2a-1}) \cup \{u_k : k = 1,2,\dots,\frac{a-1}{2}\} \cup \{z_j : j = 1,2,\dots,c - \frac{7a-3}{2}\}$ is a $\tilde{\gamma}_{con}^{p} - set$ of G. Thus $|V(G)| = (a) + (2a-1) + \left(\frac{a-1}{2}\right) + \left(c - \frac{7a-3}{2} + 1\right) = c + 1 = n$, $\gamma(G) = |A| = (a-1) + (1) = a$, $\gamma_p(G) = |B| = (a) + \left(\frac{a-1}{2}\right) = \frac{3a-1}{2} = b$, $\tilde{\gamma}_{con}^{p}(G) = |C| = (a) + (2a-1) + \left(\frac{a-1}{2}\right) + c - \frac{7a-3}{2} = c.$

This proves the assertion. ■

Corollary 2.3 The difference $\tilde{\gamma}_{con}^{p}(G) - \gamma_p(G)$ can be made arbitrarily large.

Proof: Let n be a positive integer. By Theorem 2.2, there exists a connected graph G such that $\tilde{\gamma}_{con}^{p}(G) = n + 1$ and $\gamma_p(G) = 1$. Thus $\tilde{\gamma}_{con}^{p}(G) - \gamma_p(G) = (n+1) - 1 = n$, showing that $\tilde{\gamma}_{con}^{p}(G) - \gamma_p(G)$ can be made arbitrarily large. ■

The following Remark is used to prove the next theorem.

Remark 2.4 Every complete graph is convex.

The next result is the characterization of a dominating set with a perfect outer-convex domination number of one.

Theorem 2.5 Let G be a nontrivial connected graph. Then $\tilde{\gamma}_{con}^{p}(G) = 1$ if and only if G is a complete graph.

Proof: Suppose that $\tilde{\gamma}_{con}^{p}(G) = 1$. Let S ={v} be a $\tilde{\gamma}_{con}^{p} - set$ in G. Then $V(G)\backslash S$ is convex by definition. Suppose that there exists distinct vertices $r, s \in V(G) \setminus S$ such that $rs \notin E(G)$. Since S is a dominating set, $rv, vs \in E(G)$. Thus, $r, v, s \in I_G[r,s] \subseteq I_G[V(G) \setminus S]$. Since $v \notin V(G) \setminus S$, it follows that $I_G[V(G) \setminus S] \neq V(G) \setminus S$ contrary to our assumption that $V(G) \setminus S$ is a convex set. Thus, for any $r, s \in V(G) \setminus S, rs \in E(G)$, that is, $\langle V(G)\backslash S \rangle$ is complete. Hence, G = $\langle S \rangle + \langle$V(G)\S$\rangle$ is a complete graph.

For the converse, suppose that G is a complete graph. Let $v \in V(G)$. Then $S = \{v\}$ is a dominating set of G. Since each vertex of $V(G)\backslash S$ is dominated by exactly one vertex in S, it follows that S is a perfect dominating set. Since G is complete, it follows that $\langle V(G)\backslash S \rangle$ is also complete. By Remark 2.4, $\langle V(G)\backslash S \rangle$ is convex, that is $V(G)\backslash S$ is convex. Since $V(G)\backslash S$ is convex, S is an outer-convex dominating set of G. Thus, S is a perfect outer-convex dominating set of G. Hence, $\tilde{\gamma}_{con}^{p} = 1$. ■

The following Remark is used to prove the next theorem.

Remark 2.6 If G and H are complete graphs then $G + H$ is also complete.

Let G and H be graphs of order m and n, respectively. The corona of two graphs G and H is the graph $G \circ H$ obtained by taking one copy of G and m copies of H, and then joining the ith vertex of G to every vertex of the ith copy of H. The join of vertex v of G and a copy Hv of H in the corona of G and H is denoted by $+Hv$.

The next result is the characterization of a perfect outer-convex dominating set in the corona of two graphs.

Theorem 2.7 Let G be a connected graph and H be a complete graph. Then a subset S of $V(G \circ H)$ is a perfect outer-convex dominating set in $G \circ H$ if and only if one of the following statements is satisfied:

(i) $S = \bigcup_{x \in V(G)} S_x$ where S_x is a minimum dominating set of H^x for each $x \in V(G)$ or

(ii) $S = V(G) \cup \left(\bigcup_{y \in V(G)\backslash\{x\}} V(H^y)\right)$

Proof: Suppose S is a perfect outer-convex dominating set of $G \circ H$. Then S is a perfect dominating set and $V(G)\backslash S$ is convex set. Since H is complete, by Remark 2.6, $\langle V(x + H^x) \rangle$ is complete for every $x \in V(G)$. Now suppose that S_x is a minimum dominating set of H. Then $\langle V(x + H^x)\backslash S_x \rangle$ is complete. By Remark 2.4, $\langle V(x + H^x)\backslash S_x \rangle$ is convex, that is $V(x + H^x)\backslash S_x$ is a convex set. By Theorem 2.5, S_x is a minimum perfect outer-convex dominating set of $V(x + H^x)$. Hence $S = \bigcup_{x \in V(G)} S_x$ is a perfect outer-convex dominating set of $G \circ H$ (see Fig. 6). This shows statement (i). By similar arguments used in statement (i), $S = V(G) \cup \left(\bigcup_{y \in V(G)\backslash\{x\}} V(H^y)\right)$ is a perfect outer-convex dominating set of $G \circ H$, showing statement (ii).

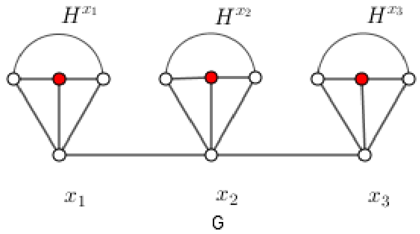

Figure 6: A Graph with a Perfect Outer-convex Dominating Set

For the converse, suppose statement (i) or (ii) holds. Clearly that $V(G \circ H)\backslash S \subseteq I_{G\circ H}[V(G \circ H)\backslash S$.Consider first that statement (i) holds. To show that $VG\circ H|\mathcal{S}$ is convex, pick two distinct vertices $u, v \in I_{G\circ H}[V(G \circ H)\backslash S]$. If $u, v \in V(G)$, then $u, v \in V(G \circ H)\backslash S$. Now, suppose $u, v \in V(H)$. Since H is complete, there exists $z \in V(H)$ such that {z} is a minimum dominating set. Thus, $u, v \in V(H)\backslash S$, implying that $u, v \in V(G \circ H)\backslash S$. Suppose that $u \in V(G)$ and $v \in V(H)$. Clearly, that $u \in V(G \circ H)\backslash S$ and $v \in V(G \circ H)\backslash S$ by previous argument. Thus $I_{G\circ H}[V(G \circ H)\backslash S] \subseteq V(G \circ H)\backslash S$. Consequently, $V(G \circ H)\backslash S = I_{G\circ H}[V(G \circ H)\backslash S]$. Hence $V(G \circ H)\backslash S$ is convex and by Theorem 2.5, S is perfect dominating set, that is, S is a perfect outer-convex dominating set. Next, consider statement (ii). Since $\langle V(H^x)\rangle$ is complete , it follows that $V(H^x)$ is convex and since every vertex of $V(H^x)$ is dominated by exactly one element in S, it follows that S is a perfect outer-convex dominating set.■

The next result is an immediate consequence of Theorem 2.7.

Corollary 2.8 Let G be a connected graph and H be a complete graph. Then $\tilde{\gamma}^{p}_{con}(G \circ H) = |V(G)|$ **.**

III. CONCLUSION

The perfect outer-convex dominating set is a new defined domination parameter which needs more intensive studies. In this paper, we limit our results on the characterization of this domination parameter in the corona of two graphs. Further research must be conducted on the characterization of other binary operations of two graphs. One possible application of this domination parameter is on the security system. This can be a good encryption method in maintaining the confidentiality of data in a digital communication.

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