International Journal of Latest Engineering Research and Applications (IJLERA) ISSN: 2455-7137

Volume –04, Issue – 10, October 2019, PP – 15-19

Application and Exploration of Case Teaching Method in Function Distribution of Random Variables in Probability Theory

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Abstract: Based on the traditional distribution function method of solving the distribution of a function of random variables, more general new method is given. It is simple to solve and has a wide range of applications. The application of the new method is described in detail with some typical examples.

Keywords: Continuous random variables, probability density, piecewise monotone

Probability theory is a basic course for statistics majors，which has always been a difficult course for students to learn.The distribution of functions of continuous random variables is a difficult point for students to learn.

Let ξ be a continuous random variable with density function $p_{\xi}(x)$,then $\eta = f(\xi)$,where $f(x)$ is a real-valued function defined on the real line, is a random variable as well.

One of the usual way to obtain probability density of $\eta = f(\xi)$ is the distribution function method, which can be divided into four steps[1]:

(ⅰ) transform the event $\{\eta \le y\}$ to $\{f(\xi) \le y\}$;

(ⅱ) transform the event $\{f(\xi) \le y\}$ to $\{\xi \in I_y\}$, where $I_y = \{x : f(x) \le y\}$;

(ⅲ) calculate the probability $p\{\xi \in I_y\}$, which is the distibution function $F_{\eta}(y)$ of η;

(ⅳ) obtain probability distribution function of η by $p_{\eta}(y) = F'_{\eta}(y)$.

If $y = f(x)$ is piecewise strictly monotone in disjoint intervals, we can directly derive the density function of the $f(\xi)$ by the following theorem.

The distribution function method can be used to prove the following theorem, where the funtion $f(x)$ is piecewise strictly monotone.

Theorem[2] If $y = f(x)$ is piecewise strictly monotone in disjoint intervals $I_1, I_2, \cdots$, which is a partition of Ω, and its inverse $h_i(y)$ in the $i-th$ interval I_i is continuously differentiable. Then $\eta = f(\xi)$ is a continuous random variable , whose density function is

$$p_\eta(y) = \begin{cases} \sum_i p_\xi(h_i(y))\left|h_i^{/}(y)\right|, & y \in \textit{the definition domain of each } h_i; \\ 0, & \textit{otherwise}. \end{cases}$$

, Where $p_\xi(x)$ is the density of ξ.

Proof. Let $F_\eta(y)$ is the distribution function of η. Observe that $\{f(\xi) \le y\} = \{\xi \in \sum_i E_i(y)\}$, where $E_i(y)$ is the set of x in I_i such that $f(x) \le y$. By the distribution function method we obtain

$$F_\eta(y) = p\{\eta \le y\} = p\{\xi \in \sum_i E_i(y)\} = \sum_i \int_{E_i(y)} p(x)dx$$

$$= \sum_i \int_{-\infty}^{y} p_\xi(h_i(x))\left|h_i^{/}(x)\right|dx = \int_{-\infty}^{y} \sum_i p_\xi(h_i(x))\left|h_i^{/}(x)\right|dx,$$

So, $$p_\eta(y) = F_\eta^{/}(y) = p_\eta(y) = \begin{cases} \sum_i p_\xi(h_i(y))\left|h_i^{/}(y)\right|, & y \in \textit{the definition domain of each } h_i; \\ 0, & \textit{otherwise}. \end{cases}$$

This completes the proof of the theorem.

However, Students will have many problems in solving concrete problems by the above theorem.

Example 1. Let $\xi \sim N(0,1)$ and $\eta = \begin{cases} \xi & ,|\xi| \le 1 \\ -\xi & ,|\xi| > 1 \end{cases}$, calculate the density function of η.

If we solve the problem with distribution function method, we have the following solving process and steps.

Solution. Let $F_\eta(y)$ is the distribution function of η, then

$$F_\eta(y) = p\{\eta < y \,|\, |\xi| \le 1\} p\{|\xi| \le 1\} + p\{\eta < y \,|\, |\xi| > 1\} p\{|\xi| > 1\}$$

$$= \begin{cases} p\{\xi < -1\} + p\{\xi > -y\} = \Phi(-1) + 1 - \Phi(-y) & , y \le -1 \\ p\{-1 \le \xi < y\} = \Phi(y) - \Phi(-1) & , -1 < y < 1, \\ p\{-y < \xi < -1\} + p\{\xi > 1\} = \Phi(-1) - \Phi(-y) + 1 - \Phi(1) & , y \ge 1 \end{cases}$$

$p_\eta(y) = F_\eta^{/}(y) = \varphi(y)$, where $\Phi(y)$ and $\varphi(y)$ are the density function and distribution function of ξ

respectively. So $\eta \sim N(0,1)$.

Some students use the above theorem to solve this problem, but the steps are as follows.

Solution. Let $y = f(x) = \begin{cases} x, & |x| < 1 \\ -x, & |x| \ge 1 \end{cases} = \begin{cases} x, & |x| < 1 \\ -x, & x \le -1, \\ x, & x \ge 1 \end{cases}$ which is piecewise strictly monotone in

$I_1 = (-\infty, -1], I_2 = (-1,1), I_3 = [1, +\infty)$.

Its inverse is $x = h(y) = f^{-1}(y) = \begin{cases} y, & |y| < 1 \\ -y, & y \le -1, \\ y, & y \ge 1 \end{cases}$ hence,

$p_\eta(y) = \sum_i p_\xi(h_i(y))\left|h_i^{/}(y)\right| = \varphi(y)\cdot 1 + \varphi(-y)\cdot 1 + \varphi(y)\cdot 1 = 3\varphi(y)$,which is fault obviously.

This example shows that the students do not have a deep understanding of Theorem1. In other words, students do not have a thorough understanding of piecewise functions.

If we use this theorem to solve the above problem, the correct solving process is as follows.

Solution. Let

$y = f(x) = \begin{cases} x, & |x| < 1 \\ -x, & |x| > 1 \end{cases} = \begin{cases} x, & |x| < 1 \\ -x, & x \le -1, \\ -x, & x \ge 1 \end{cases}$ which is piecewise strictly monotone in

$I_1 = (-\infty, -1], I_2 = (-1,1), I_3 = [1, +\infty)$.

Its inverse is $x = h(y) = f^{-1}(y) = \begin{cases} y, & |y| < 1 \\ -y, & y \le -1, \\ -y, & y \ge 1 \end{cases}$ hence, by Theroem1,

$$p_\eta(y) = \begin{cases} \varphi(y)\cdot 1, & |y| < 1 \\ \varphi(-y)\cdot 1, & y \le -1 = \varphi(y) \\ \varphi(y)\cdot 1, & y \ge 1 \end{cases}$$

What puzzles students is when to add and when to segment. In order to solve this kind of problem more accurately, we have the following theorem.

Theorem 2. Let $p_\xi(x) = \begin{cases} p(x), & a < x < b \\ 0, & otherwise \end{cases}$ is the density function of ξ and $(a,b) = \bigcup_{i=1}^{n} I_i$,where

$I_i = (x_{i-1}, x_i)$, $x_0 = a, x_n = b, x_i < x_j, i = 1,2,\cdots n$. Rank $f(x_0)$, $f(x_1), f(x_2), \cdots, f(x_n)$ from small to large as $y_1 \le y_2 \le \cdots y_{n+1}$ and let $y_{(1)}, y_{(2)}, \cdots, y_{(k)}$ are the different K values where

$y_{(1)} < y_{(2)} < \cdots < y_{(k)}$. If $y = f(x)$ is piecewise strictly monotone in disjoint intervals $I_1, I_2, \cdots I_n$ and its inverse $h_i(y)$ in the $i-th$ interval is continuously differentiable. For any $i \in \{1,2,\cdots k-1\}$, use $N(i)$ denotes the number of functions whose domain contains intervals $(y_{(i)}, y_{(i+1)})$ in the above n inverse functions and let the $N(i)$ functions as $h_{i_1}(y), h_{i_2}(y), \cdots, h_{i_{N(i)}}(y)$ .Then $\eta = f(\xi)$ is a continuous random variable ,

whose density is $p_\eta(y) = \begin{cases} \sum_{r=1}^{N(i)} p_\xi(h_{i_r}(y)) \left| h_{i_r}{}^{/}(y) \right|, & y \in (y_{(i)}, y_{(i+1)}), i = 1,2,\cdots,k-1 \\ 0, & otherwise. \end{cases}$.

Proof. Let $F_\eta(y)$ is the distribution function of η, we have

$$F_\eta(y) = p\{\eta \le y\}$$

$$= p\{f(\xi) \le y\} = p\{\bigcup_l \{\alpha_{i_l} \le \xi \le h_{i_l}(y)\} \cup \bigcup_m \{h_{i_l}(y) \le \xi \le \alpha_{i_m}\} \cup \bigcup_{s,t} \{\alpha_{i_s} \le \xi \le \alpha_{i_t}\}\}$$

, where $\alpha_{i_l}, \alpha_{i_m}, \alpha_{i_s}, \alpha_{i_t}$ are taken from the interval end points a , b

and the set of minimum points of the function $y = f(x)$ in an interval (a,b).

Furthermore, $y = f(x)$ is monotonous in $(\alpha_{i_l}, h_{i_l}(y)), (h_{i_l}(y), \alpha_{i_m})$ and $(\alpha_{i_s}, \alpha_{i_t})$, so

$$F_\eta(y) = \sum_l p\{\alpha_{i_l} \le \xi \le h_{i_l}(y)\} + \sum_m p\{h_{i_m}(y) \le \xi \le \alpha_{i_m}\} + \sum_{s,t} p\{\alpha_{i_s} \le \xi \le \alpha_{i_t}\}$$

$$= \sum_l \int_{\alpha_{i_l}}^{h_{i_l}(y)} p_\xi(x)dx + \sum_m \int_{h_{i_l}(y)}^{\alpha_{i_m}} p_\xi(x)dx + \sum_{s,t} \int_{\alpha_{i_s}}^{\alpha_{i_t}} p_\xi(x)dx,$$

then $p_\eta(y) = F_\eta{}^{/}(y) = \sum_l p_\xi(h_{i_l}(y)) \left| h_{i_l}{}^{/}(y) \right| + \sum_m p_\xi(h_{i_m}(y)) \left| h_{i_m}{}^{/}(y) \right| = \sum_{r=1}^{N(i)} p_\xi(h_{i_r}(y)) \left| h_{i_r}{}^{/}(y) \right|,$

where $h_{i_1}(y), h_{i_2}(y), \cdots, h_{i_{N(i)}}(y)$ are the $N(i)$ inverse functions of $y = f(x)$ in the definition field contained $(y_{(i)}, y_{(i+1)})$. □

The example 1 is the application of Theorem 2, Meanwhile, using Theorem 2, students can clearly know when to add and when to segment.

Let analyse another example,

Example2. $p_{\xi}(x)=\begin{cases}2x, & 0<x<1\\ 0, & otherwise\end{cases}$ is the density function of ξ . Calculate the distribution of $\eta=(\xi-\frac{2}{3})^2$.

Solution. From $y=(x-\frac{2}{3})^2$, corresponding to $(0,\frac{1}{9})$ there are two inverse functions $h_1(y)=\frac{2}{3}-\sqrt{y}$ and $h_2(y)=\frac{2}{3}+\sqrt{y}$. Corresponding to $(\frac{1}{9},\frac{4}{9})$, there is only one inverse function $h_3(y)=\frac{2}{3}-\sqrt{y}$.

By using Theorem 2 , we can easily have

$$p_{\eta}(y)=\begin{cases}p_{\xi}(h_1(y))\left|h_1'(y)\right|+p_{\xi}(h_2(y))\left|h_2'(y)\right|, & 0<y<\frac{1}{9}\\ p_{\xi}(h_3(y))\left|h_3'(y)\right|, & \frac{1}{9}<y<\frac{4}{9}\\ 0, & otherwise\end{cases}=\begin{cases}\frac{4}{3\sqrt{y}}, & 0<y<\frac{1}{9}\\ \frac{2}{3\sqrt{y}}-1, & \frac{1}{9}<y<\frac{4}{9}\\ 0, & otherwise\end{cases}.$$

Based on the well-known distribution function method, this paper studies and explores a new wider method for solving the distribution of functions of continuous random variables . This method has a wider range of applicability and more generality.

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