

## Motivating Students with Dynamic Modeling

Dr. Shinemin Lin

Department of Mathematics, Savannah State University, 3219 College Street, Savannah GA USA

**Abstract:** Dr. Winkel, director of SIMIODE promotes using computer modeling to teach Differential Equations. Dr. Panoff, director of SHORDOR.COM promotes using computer modeling to unzipped student's potential. I had a chance to attend both SIMIODE and SHORDOR workshops. I learned how to develop models that can be used in classroom and how to use models to teach classes. I used Dr. Winkel M and M model at all my classes. Different level students have different reports. I started using modeling to motivate student at all level classes. I used Dr. Panoff Have = Had + change model to inspire student working on undergraduate research in the topic of dynamic modeling of real-world cases. Since all students have different background. Dynamic modeling is the best way to encourage students working on interdisciplinary field. Differential Equations are the best way to describe the change of any phenomena. However differential equation is a spooky term for many students. If we use difference quotient that students learned in College Algebra to approximate derivative, then even lower level students can enjoy differential equation models. At this presentation, I will introduce some models that I used in my classes.

**Keywords:** Dynamic Modeling, SIMIODE, SHORDOR, M and M model, Cinderella Model

### Introduction

Many students complain that those mathematics skills they learned in classroom have nothing to do with their career after graduation. Many mathematicians tried hard to develop projects that can link classroom mathematics to real life issues. Many real-world situations are related to the change of rate of a quantity that is proportional to the amount of current amount or other quantity. Such might be the case for a population of people, rabbits, or bacteria. When money is compounded continuously, the rate of change of the amount is also proportional to the amount present. For a radioactive element, the amount of radioactive decays at a rate proportional to the amount present. All those situations can be modeled by differential equations. However, differential equations are very scary terms to many students. "Difference quotient" is more user-friendly term to students since it is the concept students learned in high school mathematics, and College Algebra classes. There are many software for dynamic modeling. Excel is one that every student already knows basic Excel skills. At this paper I will use discrete different quotient to approximate continuous differential equations and use Excel instead of other modeling software.

### Model and Analysis

#### 1. Steps of the Modeling

Keep in mind that there is no correct model. But there is always a better model. Depending on personal background, one can always create a mathematical model initially. Later one can modify its initial model and get a better realistic model. Therefore, the modeling process is cyclic and closely parallels in scientific method. The process are as following:

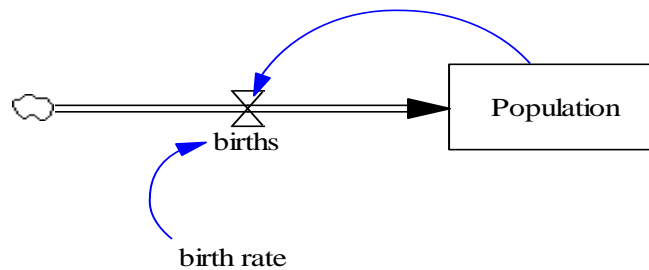
- a) Analyze the problem: At this step, students need to identify major variables and the relationship between them. It is better to draw a diagram to show the model. Students can download shareware Vensim for education. Vensim is a modeling software. I only expect use its drawing tool.
- b) Formulate the model: From the diagram, students can find out the equation between variables. Write approximation different equations.
- c) Use Excel to get graphic solutions
- d) Verify and interpret the model solution

- e) Add more conditions and go through step a), b), c) and d) to revise the model

## 2. Population Models

As an example of creating models, I shall start the simple population model.

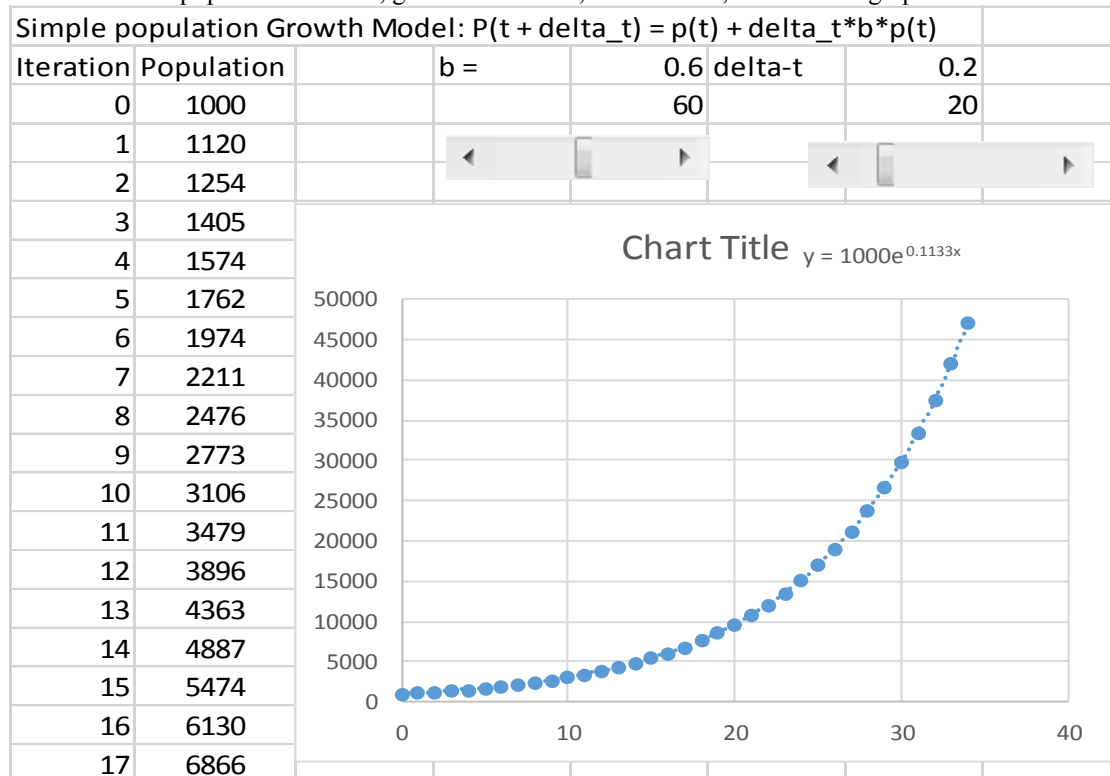
This model is a study of population growth based on a growth factor over time. The easiest way to visualize a population growth model is to assume that some proportion of the population reproduces during each time step and some proportion dies. The difference between the two is the growth factor. This model simulates a simplified model where the population increases is proportional to the current population with proportional growth constant b.



From the diagram, we know the change of the population is birth rate \* current population.

The mathematics formula is  $\frac{dP}{dt} = b * P$ , where  $\frac{dP}{dt}$  is the change of population and b is the growth constant. We can use average rate of change to approximate instantaneous rate of change. Therefore we have difference equation  $\frac{P(t+\Delta t) - P(t)}{\Delta t} = b * P(t)$ , which is the equation that we enter to Excel.

Assume initial population is 1000, growth rate is 0.6, and  $\Delta t = 0.2$ , we have the graphic solution:



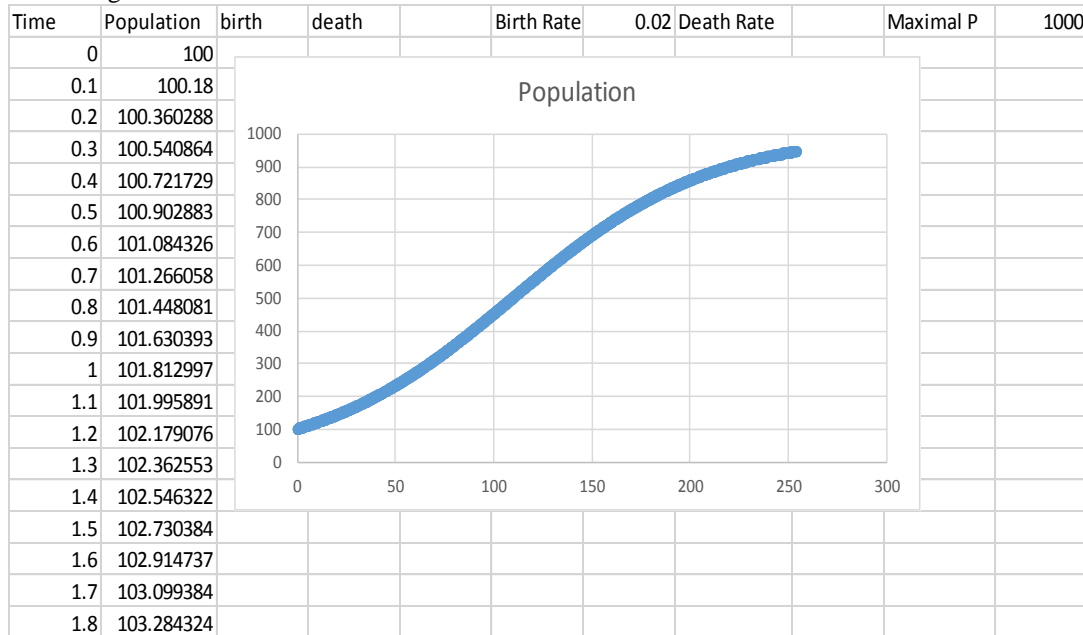
Certainly this model is not realistic since the population can grow infinitely. We can revise our model by adding competition for limit resources. If the resources of a community can only support maximal population M, the model becomes Logistics Growth Model. Initially population growth rate is the birth rate. When population is approaching the limit M, the growth rate become near to zero. We can use the equation

birth rate  $b * (1 - \frac{P}{M})$  to represent population growth rate and the equation of the change of population is

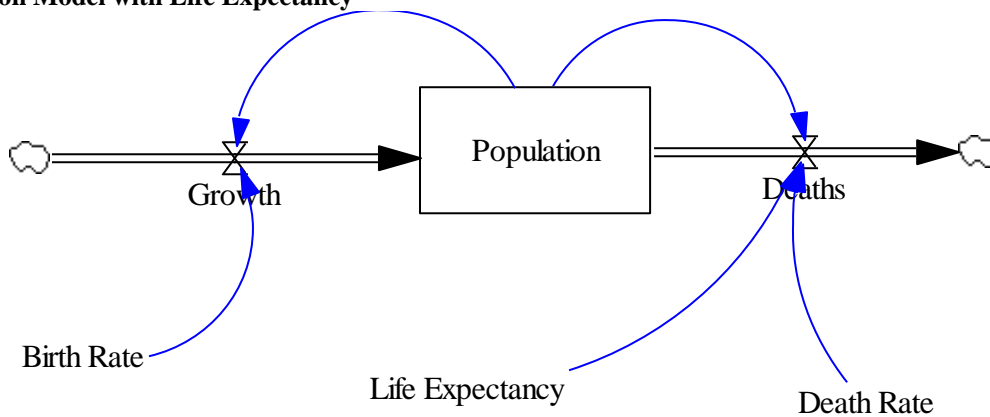
$$\frac{dp(t)}{dt} = b * (1 - \frac{P(t)}{M}) * p(t).$$

The approximation formula becomes  $\frac{P(t+\Delta t) - p(t)}{\Delta t} = b * (1 - \frac{P(t)}{M}) * P(t),$

With initial population 100, birth rate 0.02 and maximal capacity M = 1000, The following Excel solution tells it is similar to Logistic Growth model.



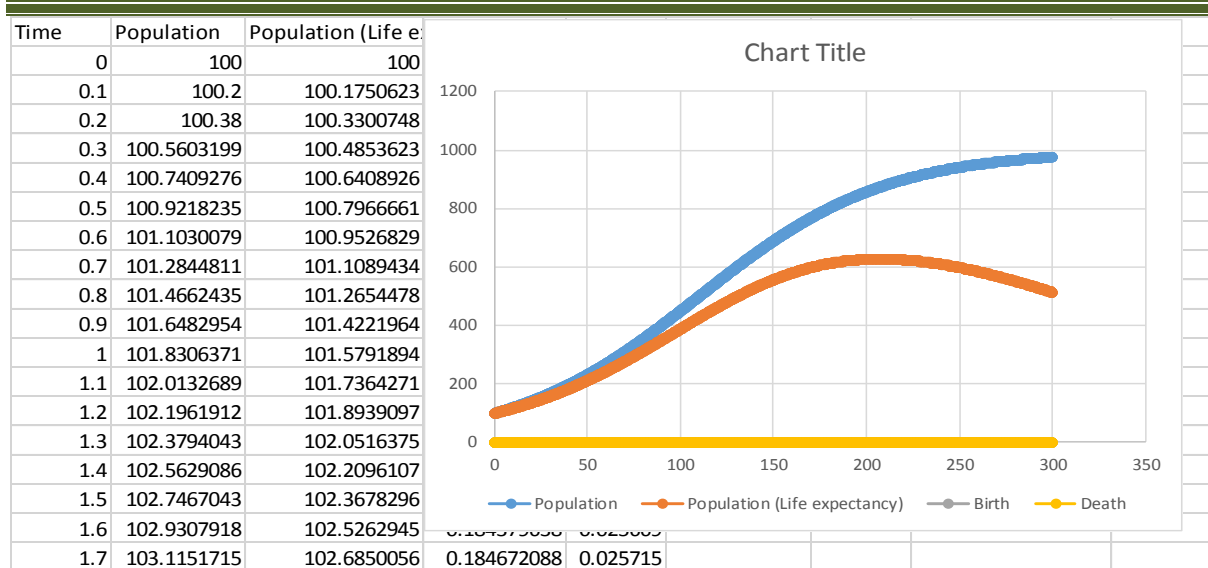
**Population Model with Life Expectancy**



Life expectancy will change population growth. If life expectancy is short, then population growth become small. If life expectance is long, then population growth is high. Life expectancy may affect growth rate.

Growth Rate = Birth rate -  $\frac{1}{1+x^2}$  where x is the life expectancy. When x = 0, there is no population growth. When x is very large, the population growth rate is very close to birth rate.

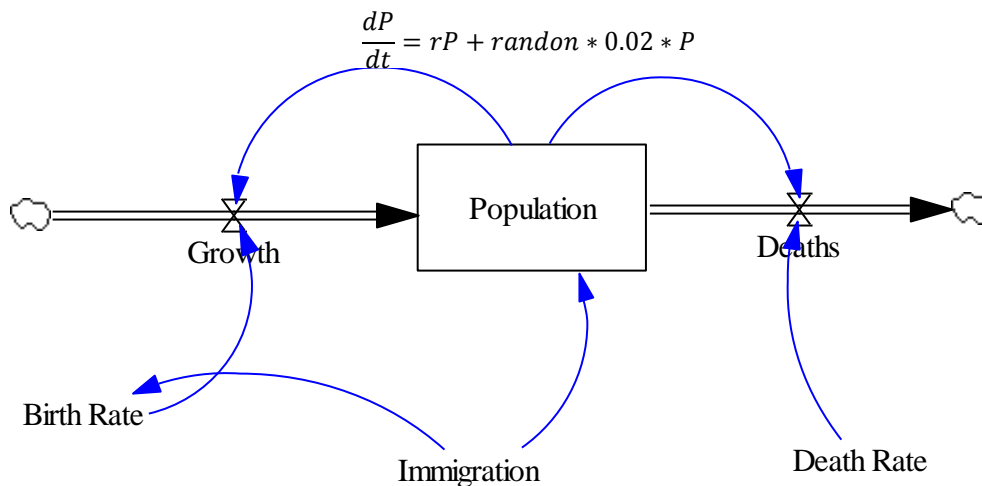
The Excel Worksheet shows the population growth with maximal capacity 1000, and life expectancy 20 assuming initial population 100, birth rate 0.02



The growth rate here is only our assumption. For real growth rate that related to life expectancy, we need to collect real data and find out.

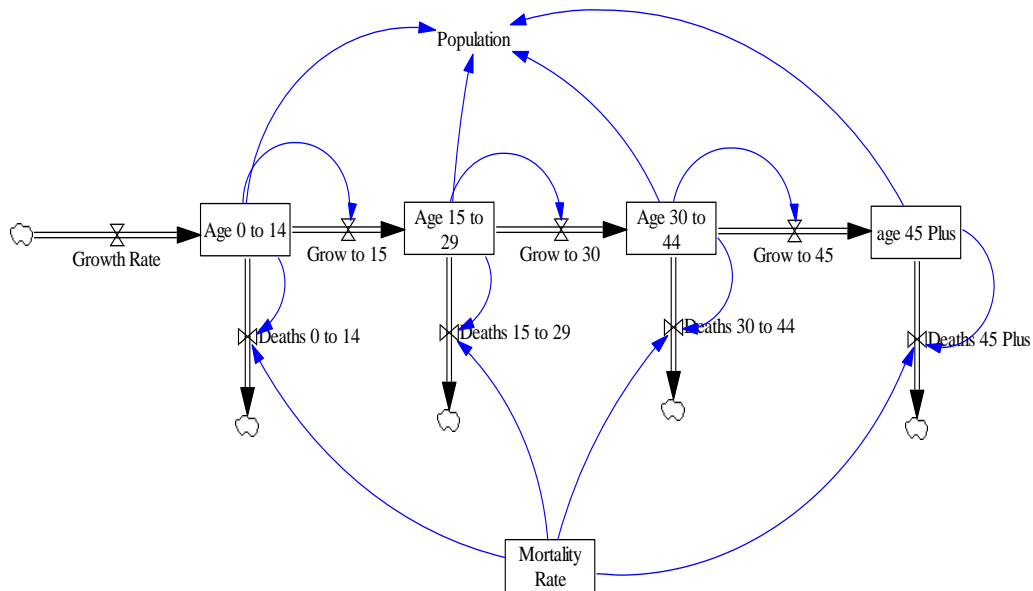
**Population Model with Immigration**

Assume that government allows at most 2% of current population immigrants to the community every period. We assume Birth rate is  $b$ , Population is  $P$ , and Death rate is  $d$ . We don't know exactly how many immigrants will move in the community. We use  $random * 0.02 * P$  to estimate immigration population per period. The population growth rate,  $r = b - d$ , and the change of population is



**Population Model with Different aged groups**

All previous model is not realistic, because we assume all people have the same fertility rate and death rate. To make our model better, we divide population into 4 groups by their ages, P15, P30, P45, and P60, where P15 is the group of people who are younger than 15 years old, P30 is the group of people whose ages are between 15 and 30, P45 is the group of people whose ages are between 30 and 45, and P60 are the people whose ages are older than 45. People in each group have different death rate, fertility rate and life expectancy. Infant mortality is counted death rate in the group P15.



The sequence of population models we introduce above is to let students know that we can always refine our model and get a better one.

### 3. Dr. Bob Panoff’s Universal Formula Have = Had + Change Model

Way back to 20 century I learned this universal formula Have = Had + change from Dr. Bob Panoff, CEO of Shodor.org. This formula can be used at most scenarios in real world. When we want to buy a house (Have), we need to save down Payment (had) and pay monthly payment (change). When we want to learn new skills (Have), we need to satisfy prerequisite (Had) and gain new skills (change).

If we use mathematics sentence to interpret this formula, we have

$$Q(t + h) = Q(t) + \Delta Q$$

But the Change is proportion to current background and the time unit. We can write the equation as

$$Q(t + h) = Q(t) + k*Q(t)*h \dots\dots\dots (a)$$

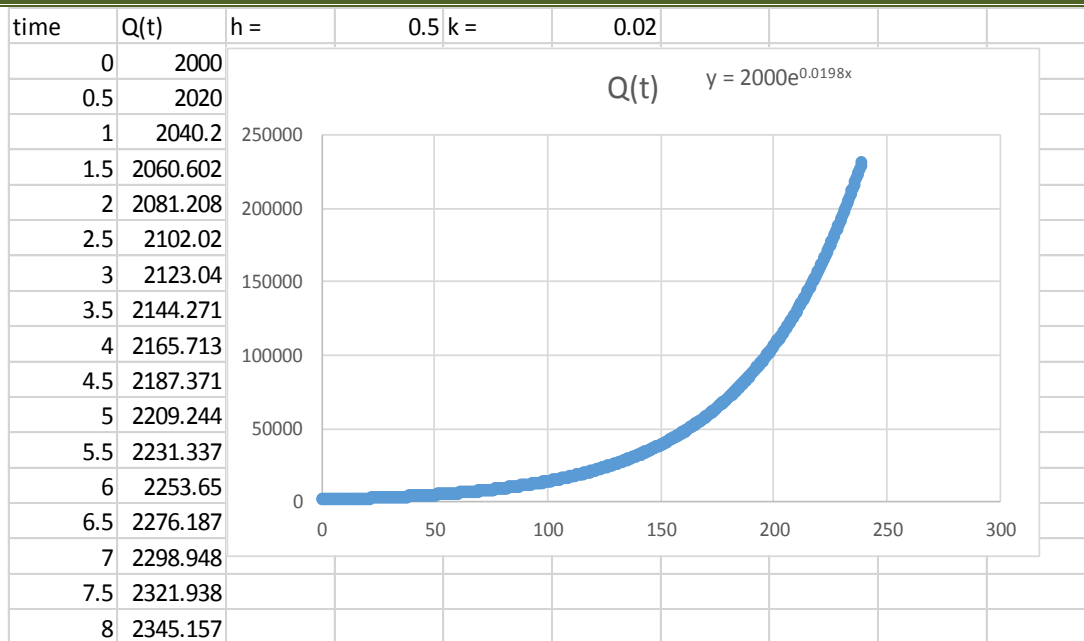
and  $\frac{Q(t+h)-Q(t)}{h} = k*Q(t) \dots\dots\dots (b)$

When time period h approaches 0, we have an exponential model differential equation

$$\frac{dQ(t)}{dt} = k* Q(t) \dots\dots\dots (c)$$

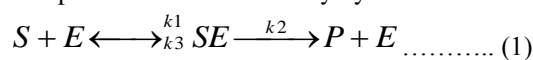
and we can have a solution  $Q(t) = Q(0)*e^{kt} \dots\dots\dots (d)$

This is the example of typical exponential growth and decay model or balance of Compounding Interest when Compound continually at College Algebra or Precalculus classes. Calculus students can solve equation (C) using separate variables method to get algebraic solution (d). Lower level students can use Scatter Plot to explore solution (d)

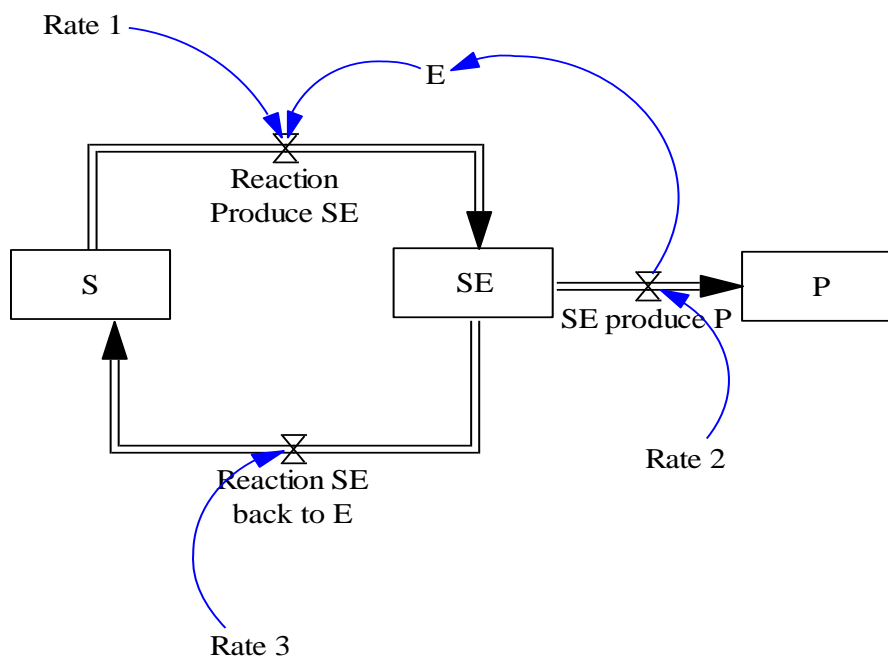


**4. Enzyme Kinetics: Basic Enzyme Reaction**

One of the most basic enzymatic reactions, first found by Michaelis and Menten (1913), involves a substrates S reacting with an enzyme E to form a complex SE which in turn is converted into a product P and the enzyme. We can represent this schematically by



Here k1, k2 and k3 are constant parameters associated with the rates of reaction. The Law of Mass Action says that the rate of a reaction is proportional to the product of the concentrations of the reactants. Let us denote the concentrations of the reactants in (1) by lower case letters s = [S], e = [E], c = [SE], p = [P], where [ ] traditionally denotes concentration.



The change of s, e, c and p are as following nonlinear differential equations

$$\frac{ds}{dt} = -k_1es + k_3c, (2)$$

$$\frac{de}{dt} = -k_1 e s + (k_2 + k_3)c, (3)$$

$$\frac{dc}{dt} = k_1 e s - (k_2 + k_3)c, (4) \text{ and}$$

$$\frac{dp}{dt} = k_2 c (5)$$

Equation (2) is simply the statement that the rate of change of the concentration [S] is made up of a loss rate proportional to [S][E] and a gain rate proportional to [SE]. Suppose the initial conditions are given by

$$s(0) = s_0, e(0) = e_0, c(0) = 0 \text{ and } p(0) = 0 (6)$$

The solutions of equations (2), (3), (4), (5) and (6) then give the concentrations and hence the rates of the reactions, as functions of time.

Adding the equation (3) add the equation (4), we have

$$\frac{de}{dt} + \frac{dc}{dt} = 0 \Rightarrow e(t) + c(t) = e_0$$

$$e(t) = e_0 - c(t)$$

Equation (2) becomes  $\frac{ds}{dt} = -k_1(e_0 - c(t))s(t) + k_3c = -k_1e_0s(t) + (k_1s + k_3)c(t)$ .

With this, the system of differential equation reduces to only three, for s, c and p, namely

$$\frac{ds}{dt} = -k_1e_0s + (k_1s + k_3)c, (7)$$

$$\frac{dc}{dt} = k_1e_0s - (k_1s + k_3 + k_2)c, (8)$$

$$\frac{dp}{dt} = k_2c (9)$$

With initial conditions  $s(0) = s_0, c(0) = 0$ , and  $p(0) = 0$ .

It is very complicate to solve these three simultaneous differential equations. With help of Excel we can get numerical and graphical solutions.

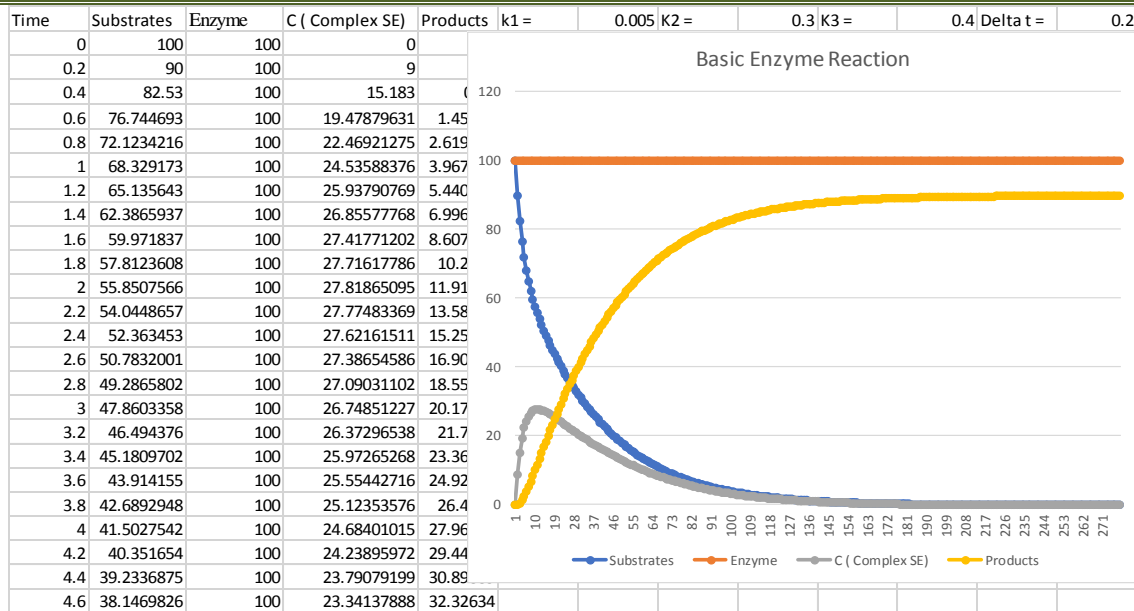
The approximation formula we entered to Excel are

$$s(t + \Delta t) = s(t) + \Delta t(-k_1e_0s + (k_1s + k_3)c)$$

$$c(t + \Delta t) = c(t) + \Delta t(k_1e_0s - (k_1s + k_3 + k_2)c)$$

$$p(t + \Delta t) = p(t) + \Delta t(k_2c)$$

The Excel solution is as following:

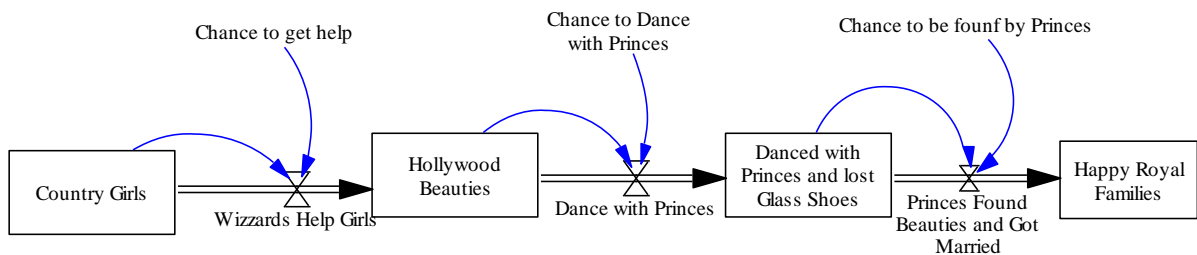


**5. Cinderella Model**

We can also create mathematics model for children stories such as Sleeping Beauty, Frog Prince, and Cinderella.

Story of Cinderella has been a most well-known fiction in little kids’ world. We can summarize the story as following: A group of country girls who wanted to marry princes. By chance some girls got wizards’ help became Hollywood beauties. By chance some of those beauties danced with princes and lost glass high hills. By chance princes found those beauties through their glass hill hills and married them.

We can draw a diagram to show this story and make a logical mathematical model.



Notations:

- G(t): Quantity of Country girls,
  - H(t): Quantity of Hollywood beauties,
  - D(t): Quantity of girls who danced with princes,
  - R(t): Quantity of royal families
  - r: chance to get help.
  - s: Chance to dance with princes.
  - t: Chance to marry princes
- The change of G(t) can be described by

$$\frac{dG}{dt} = -rG$$

Since H(t) will increase from the change of G(t) and decrease to D(t), so it can be described by

$$\frac{dH}{dt} = rG - sH$$

The same reason D(t) will gain from H(t) and lose some to R(t). The rate of change will be

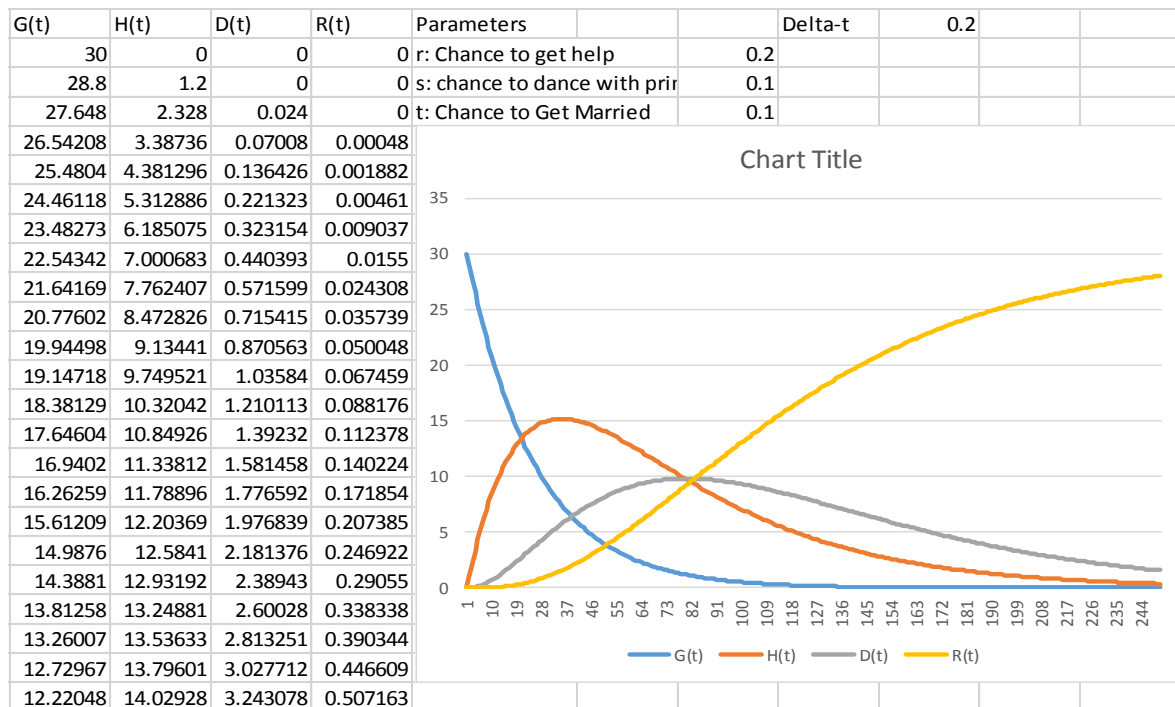


$$\frac{dD}{dt} = sH - tD$$

Finally, the R(t) is a stable quantity who only gain from D(t), so it can be described as

$$\frac{dR}{dt} = tD$$

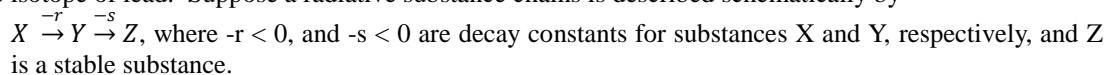
Initially if we assume  $G(0) = 30$ ,  $H(0) = 0$ ,  $D(0) = 0$ ,  $R(0) = 0$ ,  $r = 0.2$ ,  $s = 0.1$ , and  $t = 0.1$ , using Excel we can get numerical and graphic solutions of these system of differential equations.



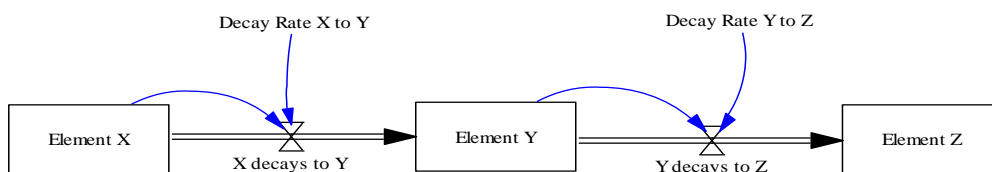
This model is no different from Radioactive Chain model: G decays to H, H decays to D, D decays to stable substance R.

### Radioactive Chains

When a substance decays by radioactivity, it usually doesn't just transmute in one step into a stable substance; rather, the first substance decays into another radiative substance, which in turn decays into a third substance, and so on. For example, the uranium decay series is  $U_{238} \rightarrow Th_{234} \rightarrow \dots \rightarrow Pb_{206}$ , where  $Pb_{206}$  is a stable isotope of lead. Suppose a radiative substance chains is described schematically by



We can draw the diagram of the radiative chain model as following:



Suppose  $x(t)$ ,  $y(t)$  and  $z(t)$  denote amounts of substances X, Y, and Z, respectively. The decay of element X is described by

$$\frac{dx}{dt} = -rx$$

Whereas the decay of element Y is

$$\frac{dy}{dt} = rx - sy$$

Since Y is gaining atoms from the decay of X and at the same time losing atoms because of its decay. Since Z is a stable element, it is simply gaining atoms from the decay of element Y:

$$\frac{dz}{dt} = -sy$$

In other words, a model of radioactive decay series for the three elements is a linear system of three first-order differentiations

$$\begin{aligned}\frac{dx}{dt} &= -rx \\ \frac{dy}{dt} &= rx - sy \\ \frac{dz}{dt} &= -sy\end{aligned}$$

Looking at those equations, the system is close to popular SIR model.

### Feedback:

The sequence of population models is to let students know they can create mathematics models base on their background. They can have simple population model that can be applied to many real-life issues. Certainly they can refine to get better models. Usually I used Have = Had + Change model at the first class of every course. The purpose is to let them know that they come to class to learn something they are not familiar. The learning of new skills is depending on their background. If they don't have good background, then they need to work extra hard to increase Change. Certainly it also tells students that they cannot forget what they learned after final test. Everything they learned will be used in the future. Enzyne Dinamic Model is an example of application of Mathematics in Biology or Chemistry. No matter what topics in any subjects, as long as one can tell the story, one should be able to draw the diagram of the story and hence create a mathematics model. Cinderella Model is to let students know we can apply a fairy tale to STEM fields. I always believe that Mathematical modeling is the best way to connect classroom mathematics to real-world issues. I use those examples to encourage students to create mathematics model whenever they have a story.

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