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Separation, Connectedness and Compactness in Binary Čech Closure spaces

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Abstract: In this paper we study mainly the separation axioms in Binary Čech Closure Spaces(BČCS) and establish the relation between them. We also introduce the concepts of connectedness and compactness in BČCS with sufficient examples.

Keywords and Phrases: Binary Čech Closure Spaces, $\breve{b}$-separation properties, $\breve{b}$-connectedness, $\breve{b}$-compactness.

AMS Subject Classification: 54A05

1 Introduction

Closure spaces were introduced by E. Čech [1] and then studied by many authors like David Niel Roth[2], Sunitha T. A.[3] etc. Čech closure spaces, is a generalisation of the concept of topological spaces. D. N. Roth and J. W. Carlson [2] studied a number of separation properties in closure spaces. W. J. Thron studied some separation properties in closure spaces. T. A. Sunitha[3] studied higher separation properties in closure spaces. P. Thangavelu and Nithanantha Jothi introduced the concept of binary topology[5]. It is a single topological structure that carries the subsets of a set X as well as the subsets of another set Y for studying the information about the ordered pair (A,B) of subsets of X and Y. Tresa Chacko and D. Susha introduced Binary Čech Closure Spaces in [11]. In this paper we introduce separation properties, connectedness and compactness in Binary Čech Closure Spaces.

The paper is organised as follows.

Section 2 contains the pre-requisites for the paper. In Section 3 we define separation properties and establish the relation between them. We also prove characterisations for $\breve{b}$-regular and $\breve{b}$-normal spaces.

Section 4 describes the concept of $\breve{b}$-connectedness, Hausdorff- Lennes condition in binary spaceand establish the relation between $\breve{b}$-connectedness and Hausdorff- Lennes condition.Feeble $\breve{b}$-connectedness is also introduced and explained with sufficient examples.

Section 5 deals with $\breve{b}$-compactness and a property of $\breve{b}$-compactness in connection with the finite intersection property.

2 Preliminaries

Definition 1 *[1] Let X be a set and $\wp(X)$ be its powerset. A function $c:\wp(X)\to\wp(X)$ is called a Čech closure operator for X if*

1. $c(\phi)=\phi$
2. $A\subseteq c(A)$
3. $c(A\cup B)=c(A)\cup c(B), \forall A,B\subseteq X$

Then (X,c) is called Čech closure space or simply closure space.

If in addition

4. $c(c(A))=c(A), \forall A\subseteq X,$

the space (X,c) is called a Kuratowski (topological) space.

If further

5. for any family of subsets of $X, \{A_i\}_{(i\in I)}, c(\cup_{i\in I} A_i) = \cup_{i\in I} c(A_i)$, the space is called a total closure space.

Definition 2 *[1] A function* $c : \wp(X) \to \wp(X)$ *is called a monotone operator for* X *if*

1. $c(\phi) = \phi$
2. $A \subseteq c(A)$
3. $A \subseteq B \Rightarrow c(A) \subseteq c(B), \forall A, B \subseteq X$

Then (X, c) is called monotone space.

Definition 3 *[5] Let* X *and* Y *be any two non-empty sets and* $\wp(X)$ *and* $\wp(Y)$ *be their power sets respectively. A binary topology from* X *to* Y *is a binary structure* $M \subseteq \wp(X) \times \wp(Y)$ *that satisfies the following axioms.*

1. (ϕ, ϕ) and $(X, Y) \in M$
2. If (A_1, B_1) and $(A_2, B_2) \in M$, then $(A_1 \cap A_2, B_1 \cap B_2) \in M$.
3. If $\{(A_\alpha, B_\alpha) : \alpha \in \Delta\}$ is a family of members of M, then $(\cup_{\alpha\in\Delta} A_\alpha, \cup_{\alpha\in\Delta} B_\alpha) \in M$.

If M is a binary topology from X to Y then the triplet (X, Y, M) is called a binary topological space and the members of M are called binary open sets. (C, D) is called binary closed if $(X \setminus C, Y \setminus D)$ is binary open.

The elements of $X \times Y$ are called the binary points of the binary topological space (X, Y, M).

Two binary points, (x_1, y_1) and (x_2, y_2) are distinct if either $x_1 \neq x_2$ or $y_1 \neq y_2$ or both. They are jointly distinct if both $x_1 \neq x_2$ and $y_1 \neq y_2$.

Let (X, Y, M) be a binary topological space and let $(x, y) \in X \times Y$. The binary open set (A, B) is called a binary neighbourhood of (x, y) if $x \in A$ and $y \in B$.

If $X = Y$ then M is called a binary topology on X and we write (X, M) as a binary space.

Note: $\wp(X)$ denotes the power set of a set X.

Definition 4 *[11] Let* X *and* Y *be two sets. A function* $\breve{b} : \wp(X) \times \wp(Y) \to \wp(X) \times \wp(Y)$ *is called a binary closure (monotone) operator if*

$$\breve{b}(\phi, \phi) = (\phi, \phi)(A, B) \subseteq \breve{b}(A, B)(A, B) \subseteq (C, D) \Rightarrow \breve{b}(A, B) \subseteq \breve{b}(C, D).$$

Then $(X, Y, \breve{b})$ is called a binary closure (monotone) space.

The binary closure operator is a binary $\breve{C}$ech closure operator if it satisfies

$$\breve{b}[(A, B) \cup (C, D)] = \breve{b}(A, B) \cup \breve{b}(C, D).$$

Definition 5 *[11] A set* $(A, B) \in \wp(X) \times \wp(Y)$ *is* $\breve{b}$*-closed if* $\breve{b}(A, B) = (A, B)$ *and a set* (C, D) *is* $\breve{b}$*-open if* $\breve{b}(X \setminus C, Y \setminus D) = (X \setminus C, Y \setminus D)$.

Proposition 1 *[11] Let* $(X, Y, \breve{b})$ *be a binary* $\breve{C}$*ech closure space. Then* (ϕ, ϕ) *and* (X, Y) *are both open and closed.*

Proposition 2 *[11] If* (X, c_1) *and* (Y, c_2) *are two* $\breve{C}$*ech closure spaces, then* $(X, Y, \breve{b})$ *where* $\breve{b} : \wp(X) \times \wp(Y) \to \wp(X) \times \wp(Y)$ *is given by* $\breve{b}(A, B) = (c_1(A), c_2(B))$, *is a binary* $\breve{C}$*ech closure*

operator.

Proposition 3 *[11] Let* $(X,Y,\breve{b})$ *be a binary* $\breve{C}$ *ech closure space. Then the set of all* $\breve{b}$ *-open sets ,i.e.* $M(\breve{b}) := \{(A,B) \mid \breve{b}(X \setminus A, Y \setminus B) = (X \setminus A, Y \setminus B)$ *is a binary topology.*

Definition 6 *Let* $(X,Y,\breve{b})$ *be a binary* $\breve{C}$ *ech closure space. Then the binary interior operator associated with* $\breve{b}$ *,* $Int_{\breve{b}}$ *is a function from* $\wp(X)\times\wp(Y)$ *to itself given by* $Int_{\breve{b}}(A,B) = (X \setminus C, Y \setminus D)$ *where* $(C,D) = \breve{b}(X \setminus A, Y \setminus B)$.

A binary set (A,B) is $\breve{b}$ -open if and only if $Int_{\breve{b}}(A,B) = (A,B)$

Proposition 4 *Given a binary* $\breve{C}$ *ech closure operator,* $\breve{b}$ *from* X *to* Y *, the function* $\breve{b}_X : \wp(X) \to \wp(X)$ *given by* $\breve{b}_X(A) = C$ *where* $\breve{b}(A,\phi) = (C,D)$ *is a* $\breve{C}$ *ech closure operator on* X *. Similarly* $\breve{b}_Y : \wp(Y) \to \wp(Y)$ *given by* $\breve{b}_Y(B) = D$ *where* $\breve{b}(\phi,B) = (C,D)$ *is a* $\breve{C}$ *ech closure operator on* Y *.*

Lemma 1 *Let* $(X,Y,\breve{b})$ *be a binary* $\breve{C}$ *ech closure operator. Then* $(\breve{b}_X(A), \breve{b}_Y(B)) \subseteq \breve{b}(A,B) \forall (A,B) \in \wp(X)\times\wp(Y)$.

3 Separation Properties

Definition 7 *Let* $(X,Y,\breve{b})$ *be a binary* $\breve{C}$ *ech closure space. It is said to be* $\breve{b}$ *-*T_0 *if for every pair of distinct binary points* (x_1,y_1) *and* $(x_2,y_2) \in X\times Y$ *, either* $(x_1,y_1) \notin \breve{b}(\{x_2\},\{y_2\})$ *or* $(x_2,y_2) \notin \breve{b}(\{x_1\},\{y_1\})$.

Proposition 5 *If* $(X,Y,\breve{b})$ *is a* $\breve{b}$ *-*T_0 *B*$\breve{C}$ *CS, then* $(X,\breve{b}_X)$ *and* $(Y,\breve{b}_Y)$ *are* T_0 $\breve{C}$ *ech closure spaces.*

Proof. Let x_1, x_2 be two distinct points in X and y_1, y_2 be two distinct points in Y.

Let $(X,\breve{b}_X)$ be not a T_0 $\breve{C}$ ech closure space.

Then there exists two distinct points $x_1, x_2 \in X$ such that $x_1 \in \breve{b}_X(\{x_2\})$ and $x_2 \in \breve{b}_X(\{x_1\})$.

Then for any $y \in Y, (x_1,y) \neq (x_2,y)$.

$x_2 \in \breve{b}_X(\{x_1\})$ and $(\breve{b}_X(\{x_1\}),\phi) \subseteq \breve{b}(\{x_1\},\{y\}) \Rightarrow (x_2,y) \in \breve{b}(\{x_1\},\{y\})$.

Similarly $(x_1,y) \in \breve{b}(\{x_2\},\{y\})$.

Hence it contradicts that $(X,Y,\breve{b})$ is a $\breve{b}$ -T_0 B$\breve{C}$ CS. So $(X,\breve{b}_X)$ is a T_0 $\breve{C}$ ech closure space.

Similarly we can prove that $(Y,\breve{b}_Y)$ is a T_0 $\breve{C}$ ech closure space.

Hence the theorem.

Remark 1 $(X,\breve{b}_X)$ *and* $(Y,\breve{b}_Y)$ *are both* T_0 $\breve{C}$ *ech closure spaces, need not imply* $(X,Y,\breve{b})$ *is a* $\breve{b}$ *-*T_0 *B*$\breve{C}$ *CS.*

Example 1 *Let* $X = \{0,1\}, Y = \{p,q\}$.

Let $\breve{b} : \wp(X)\times\wp(Y) \to \wp(X)\times\wp(Y)$ be defined as follows:

$\breve{b}(\{0\},\phi)=(\{0\},\phi)$, $\breve{b}(\{1\},\phi)=(\{0,1\},\{p\})$, $\breve{b}(\phi,\{p\})=(\phi,\{p,q\})$, $\breve{b}(\phi,\{q\})=(\{0\},\{q\})$

For all other $(A,B)\in\wp(X)\times\wp(Y)$,

$\breve{b}(A,B)=[\cup_{x\in A}\breve{b}(\{x\},\phi)]\cup[\cup_{y\in B}\breve{b}(\phi,\{y\})]$

Here X is $\breve{b}_X$ - T_0 and Y is $\breve{b}_Y$ - T_0. But $(X,Y,\breve{b})$ is not a $\breve{b}$ - T_0 B $\breve{C}$ CS, since $(\{1\},\{q\})\in\breve{b}(\{1\},\{p\})$ and $(\{1\},\{p\})\in\breve{b}(\{1\},\{q\})$

Definition 8 *A binary* $\breve{C}$ *ech closure space* $(X,Y,\breve{b})$ *is said to be* $\breve{b}$ - T_1 *if for two distinct binary points,* (x_1,y_1) *and* (x_2,y_2) *in* $X\times Y,(x_1,y_1)\notin\breve{b}\{(\{x_2\},\{y_2\})\}$ *and* $(x_2,y_2)\notin\breve{b}\{(\{x_1\},\{y_1\})\}$.

Proposition 6 *The following statements are equivalent in any binary* $\breve{C}$ *ech closure space.*

1. The space $(X,Y,\breve{b})$ is $\breve{b}$ - T_1.
2. For any binary point $(x,y)\in X\times Y$, $(\{x\},\{y\})$ is $\breve{b}$ -closed.
3. If $A\subseteq X$ and $B\subseteq Y$ are both finite sets then, (A,B) is $\breve{b}$ - closed.

Proof. $(1)\Rightarrow(2)$

Let $(X,Y,\breve{b})$ be T_1.

Assume that $(\{x\},\{y\})$ is not $\breve{b}$ -closed.

Then $\breve{b}(\{x\},\{y\})\neq(\{x\},\{y\})$ *i.e.* $\exists(x',y')[\neq(x,y)]\in X\times Y$, such that $(x',y')\in\breve{b}(\{x\},\{y\})$. This contradicts the fact that $(X,Y,\breve{b})$ is $\breve{b}$ - T_1.

$(\{x\},\{y\})$ is $\breve{b}$ -closed.

$(2)\Rightarrow(3)$

Since $\breve{b}(A_1,B_1)\cup\breve{b}(A_2,B_2)=\breve{b}(A_1\cup B_1,A_2\cup B_2)$, if A and B are finite, (A,B) is $\breve{b}$ -closed by (2).

$(3)\Rightarrow(2)$

Follows directly from (3).

$(2)\Rightarrow(1)$

If (x_1,y_1) and (x_2,y_2) are two distinct binary points in $X\times Y,(x_1,y_1)\notin\breve{b}(\{x_2\},\{y_2\})=(\{x_2\},\{y_2\})$ and $(x_2,y_2)\notin\breve{b}(\{x_1\},\{y_1\})=(\{x_1\},\{y_1\})$.

Thus $(X,Y,\breve{b})$ is $\breve{b}$ - T_1.

Remark 2 *Every* $\breve{b}$ - T_1 *space is* $\breve{b}$ T_0, *but the converse is not true.*

Example 2 *Let* $X=\{0,1\},Y=\{p,q\}$.

Let $\breve{b}:\wp(X)\times\wp(Y)\to\wp(X)\times\wp(Y)$ be defined as follows:

$\breve{b}(\{0\},\phi)=(\{0\},\phi)$, $\breve{b}(\{1\},\phi)=(\{0,1\},\phi)$, $\breve{b}(\phi,\{p\})=(\phi,\{p,q\})$, $\breve{b}(\phi,\{q\})=(\phi,\{q\})$

For all other $(A,B)\in\wp(X)\times\wp(Y)$,

$\breve{b}(A,B) = [\cup_{x\in A}\breve{b}(\{x\},\phi)] \cup [\cup_{y\in B}\breve{b}(\phi,\{y\})]$

Here $(X,Y,\breve{b})$ is $\breve{b}$-T_0, but $(X,Y,\breve{b})$ is not $\breve{b}$-T_1, since $\breve{b}(\{0\},\{p\}) \neq (\{0\},\{p\})$.

Proposition 7 *If* $(X,Y,\breve{b})$ *is a* $\breve{b}$-T_1 *B*$\breve{C}$ *CS, then* $(X,\breve{b}_X)$ *and* $(Y,\breve{b}_Y)$ *are* T_1 $\breve{C}$ *ech closure spaces.*

Proof. Let $(X,Y,\breve{b})$ be $\breve{b}$-T_1. Then for any $(x,y) \in X \times Y, \breve{b}(\{x\},\{y\}) = (\{x\},\{y\})$.

We have $(\breve{b}_X(\{x\}),\phi) \subseteq \breve{b}(\{x\},\{y\}) \Rightarrow \breve{b}_X(\{x\}) \subseteq \{x\}.i.e.\breve{b}_X(\{x\}) = \{x\}$, since we have $\{x\} \subseteq \breve{b}_X(\{x\})$.

Thus $\breve{b}_X$ is a T_1 $\breve{C}$ech closure operator. Similarly $\breve{b}_Y$ is also a T_1 $\breve{C}$ech closure operator.

Remark 3 *Converse of the above Proposition need not be true.*

Example 3 *Let* $X = \{0,1,2\}$ *and* $Y = \{p,q\}$.

Let $\breve{b} : \wp(X) \times \wp(Y) \to \wp(X) \times \wp(Y)$ be defined as follows:

$\breve{b}(\{0\},\phi) = (\{0\},\{p\}),\quad \breve{b}(\{1\},\phi) = (\{1\},\{p\}),\quad \breve{b}(\{2\},\phi) = (\{2\},\{p\}), \breve{b}(\phi,\{p\}) = (\{1\},\{p\}),$

For all other $(A,B) \in \wp(X) \times \wp(Y)$,

$\breve{b}(A,B) = [\cup_{x\in A}\breve{b}(\{x\},\phi)] \cup [\cup_{y\in B}\breve{b}(\phi,\{y\})]$

Here X is $\breve{b}_X$ - T_1 and Y is $\breve{b}_Y$ - T_1. But $(X,Y,\breve{b})$ is not a $\breve{b}$ - T_1 B $\breve{C}$ CS, since $\breve{b}(\{0\},\{p\}) \neq (\{0\},\{p\})$.

Definition 9 *A binary closure space* $(X,Y,\breve{b})$ *is said to be* $\breve{b}$*- Hausdorff if for two distinct binary points* (x_1,y_1) *and* (x_2,y_2), *there exists* $\breve{b}$*-open sets* (U_1,V_1) *and* (U_2,V_2) *of* (x_1,y_1) *and* (x_2,y_2) *respectively such that* $(U_1,V_1) \cap (U_2,V_2) = (\phi,\phi)$.

Proposition 8 *Let* $(X,Y,\breve{b})$ *is said to be* $\breve{b}$*- Hausdorff. Then* $(X,\breve{b}_X)$ *and* $(Y,\breve{b}_Y)$ *are Hausdorff* $\breve{C}$*ech closure spaces.*

Proof. Let $x_1 \neq x_2 \in X$ and $y_1 \neq y_2 \in Y$ .Then $(x_1,y_1) \neq (x_2,y_2)$. Hence there exists $\breve{b}$-open sets (U_1,V_1) and (U_2,V_2) of (x_1,y_1) and (x_2,y_2) respectively such that $(U_1,V_1) \cap (U_2,V_2) = (\phi,\phi)$. Then U_1,U_2 are $\breve{b}_X$ neighbourhoods of x_1,x_2 respectively in $(X,\breve{b}_X)$ and V_1,V_2 are $\breve{b}_Y$ neighbourhoods of y_1,y_2 respectively in $(Y,\breve{b}_Y)$. Also $U_1 \cap U_2 = \phi$ and $V_1 \cap V_2 = \phi$, showing that $(X,\breve{b}_X)$ and $(Y,\breve{b}_Y)$ are Hausdorff $\breve{C}$ech closure spaces.

Remark 4 *Every* $\breve{b}$*-Hausdorff space is* $\breve{b}$ T_1*, but the converse is not true.*

Example 4 *Let* X *and* Y *be any two infinite spaces and* $\breve{b}$ *be a Binary* $\breve{C}$*ech Closure operator defined from* X *to* Y *as follows:*

$\breve{b}(A,B) =$ A *and* B *arebothfinite*(X,B), if A *isinfinite and* B *isfinite*(A,Y), if A *isfinitean d* B *isinfinite*(X,Y), if A *and* B *arebothinfinite*

Then $(X,Y,\breve{b})$ is a $\breve{b}$ T_1 space, but not $\breve{b}$-Hausdorff, by Proposition 8, since the topology obtained

from $(X,\breve{b}_X)$, the cofinite topology is not Hausdorff, $(X,\breve{b}_X)$ is not Hausdorff[3].

Definition 10 *A binary closure space* $(X,Y,\breve{b})$ *is said to be*

1. $\breve{b}$-Urysohn space if for two distinct binary points $(x_1,y_1),(x_2,y_2)$ there exists $\breve{b}$-open sets $(U_1,V_1),(U_2,V_2)$ such that $(x_1,y_1)\in(U_1,V_1),(x_2,y_2)\in(U_2,V_2)$ and $\breve{b}(U_1,V_1)\cap\breve{b}(U_2,V_2)=(\phi,\phi)$.
2. $\breve{b}$-regular if for each binary point (x,y) and each $\breve{b}$-closed set (A,B) such that $(x,y)\notin\breve{b}(A,B)$, there exists $\breve{b}$-open sets $(U_1,V_1),(U_2,V_2)$ such that $(x,y)\in(U_1,V_1),(A,B)\subseteq(U_2,V_2)$ and $(U_1,V_1)\cap(U_2,V_2)=(\phi,\phi)$.
3. $\breve{b}$-normal if for any pair of jointly disjoint $\breve{b}$-closed sets (A_1,B_1) and (A_2,B_2), there exists jointly disjoint $\breve{b}$-open sets (U_1,V_1) and (U_2,V_2) containing (A_1,B_1) and (A_2,B_2) respectively.

Proposition 9 *A* $B\breve{C}$ *CS* $(X,Y,\breve{b})$ *is*

1. $\breve{b}$-Urysohn $\Rightarrow$ $\breve{b}$-Hausdorff
2. $\breve{b}$-regular and $\breve{b}$-T_1 $\Rightarrow$ $\breve{b}$-Hausdorff
3. $\breve{b}$-normal and $\breve{b}$-T_1 $\Rightarrow$ $\breve{b}$-regular.

Proof.

1. Let $(X,Y,\breve{b})$ be $\breve{b}$-Urysohn. Then for any two binary points $(x_1,y_1),(x_2,y_2)$ there exists binary open sets $(U_1,V_1),(U_2,V_2)$ such that $(x_1,y_1)\in(U_1,V_1),(x_2,y_2)\in(U_2,V_2)$ and $\breve{b}(U_1,V_1)\cap\breve{b}(U_2,V_2)=(\phi,\phi)$. Now if $(U_1,V_1)\cap(U_2,V_2)\neq(\phi,\phi),(U_1,V_1)\subseteq\breve{b}(U_1,V_1)$ and $(U_2,V_2)\subseteq\breve{b}(U_2,V_2)\Rightarrow\breve{b}(U_1,V_1)\cap\breve{b}(U_2,V_2)\neq(\phi,\phi)$. Hence $(U_1,V_1)\cap(U_2,V_2)=(\phi,\phi)$ and $(X,Y,\breve{b})$ is $\breve{b}$-Hausdorff.
2. Let $(X,Y,\breve{b})$ be $\breve{b}$-regular and $\breve{b}$-T_1. Then for each binary point (x,y), $(\{x\},\{y\})$ is $\breve{b}$-closed and by the definitions of $\breve{b}$-regular and $\breve{b}$-Hausdorff, the result follows.
3. The proof follows from the definitions and it is similar to the above proof.

Proposition 10 *Let* $(X,Y,\breve{b})$ *be a* $B\breve{C}$ *CS. Then it is* $\breve{b}$*-regular if and only if for any binary element* (x,y) *and any* $\breve{b}$*-open set* (G_1,G_2) *containing* (x,y)*, there exists a* $\breve{b}$*-open set* (H_1,H_2) *containing* (x,y)*, such that* $\breve{b}(H_1,H_2)\subseteq(G_1,G_2)$

Proof. If (G_1,G_2) is a $\breve{b}$-open set containing (x,y), $(X\setminus G_1,Y\setminus G_2)$ is a $\breve{b}$-closed set that does not contain (x,y). So by $\breve{b}$-regularity, there exists two $\breve{b}$-open sets (U_1,U_2) and (V_1,V_2) such that

$(x,y)\in(U_1,U_2),(X\setminus G_1,Y\setminus G_2)\subseteq(V_1,V_2)$ and $(U_1,U_2)\cap(V_1,V_2)=(\phi,\phi)$.

Then $(U_1,U_2)\subseteq(X\setminus V_1,Y\setminus V_2)$ and so $\breve{b}(U_1,U_2)\subseteq\breve{b}(X\setminus V_1,Y\setminus V_2)$.

Since (V_1,V_2) is $\breve{b}$-open, $(X\setminus V_1,Y\setminus V_2)$ is $\breve{b}$-closed and $\breve{b}(X\setminus V_1,Y\setminus V_2)=(X\setminus V_1,Y\setminus V_2)$.

Thus $\breve{b}(U_1,U_2)\subseteq(X\setminus V_1,Y\setminus V_2)$.

Now $(X \setminus G_1, Y \setminus G_2) \subseteq (V_1, V_2) \Rightarrow (X \setminus V_1, Y \setminus V_2) \subseteq (G_1, G_2) \Rightarrow \breve{b}(U_1, U_2) \subseteq (G_1, G_2)$.

So taking $(H_1, H_2) = (U_1, U_2)$ proves the theorem.

Conversely suppose that $(x, y) \in (X, Y)$ and (A, B) be a $\breve{b}$-closed set not containing (x, y). Then $(X \setminus A, Y \setminus B)$ is a $\breve{b}$-open set containing (x, y). Hence there exists a $\breve{b}$-open set (H_1, H_2) containing (x, y), such that $\breve{b}(H_1, H_2) \subseteq (X \setminus A, Y \setminus B)$.

Thus $(H_1, H_2) \subseteq \breve{b}(H_1, H_2) \subseteq (X \setminus A, Y \setminus B) \Rightarrow (A, B) \subseteq (X \setminus H_1, Y \setminus H_2)$.

Then $\breve{b}$-int-$(X \setminus H_1, Y \setminus H_2)$ is a $\breve{b}$-open set containing (A, B) and it is disjoint with (H_1, H_2).

Proposition 11 *A $B\breve{C}CS$, $(X, Y, \breve{b})$ is $\breve{b}$-normal if and only if for any $\breve{b}$-closed set (C, D) and any $\breve{b}$-open set (U, V) containing (C, D) there exists a $\breve{b}$-open set (H_1, H_2) such that $(C, D) \subseteq (H_1, H_2)$ and $\breve{b}(H_1, H_2) \subseteq (U, V)$.*

Proof. Proof is similar to that of the Proposition 10.

4 Connectedness

Definition 11 *A $B\breve{C}CS$ $(X, Y, \breve{b})$ is said to be $\breve{b}$-disconnected if there exists two binary subsets (A_1, B_1) and (A_2, B_2) such that*

$\breve{b}(A_1, B_1) \cup \breve{b}(A_2, B_2) = (X, Y)$ and $\breve{b}(A_1, B_1) \cap \breve{b}(A_2, B_2) = (\phi, \phi)$.

A space which is not $\breve{b}$-disconnected is called $\breve{b}$-connected.

Remark 5 *$(X, Y, \breve{b})$ is $\breve{b}$-disconnected neednot imply $(X, \breve{b}_X)$ and $(Y, \breve{b}_Y)$ are disconnected $\breve{C}$ech closure spaces. Also $(X, \breve{b}_X)$ and $(Y, \breve{b}_Y)$ are disconnected $\breve{C}$ech closure spaces neednot imply $(X, Y, \breve{b})$ is $\breve{b}$-disconnected.*

Example 5 *Let $X = \{1,2,3\}$ and $Y = \{e, f\}$. Let $\breve{b}$ be a Binary $\breve{C}$ech Closure operator defined between X and Y as follows.*

$\breve{b}(\{1\}, \phi) = (\{1,2\}, \phi)\breve{b}(\{2\}, \phi) = (\{1,2\}, \{f\})\breve{b}(\{3\}, \phi) = (\{3\}, Y)\breve{b}(\phi, \{e\}) = (\{2,3\}, \{e\})\breve{b}(\phi, \{f\}) = (\{1$

For all other $(A, B) \in \wp(X) \times \wp(Y)$,

$\breve{b}(A, B) = [\cup_{x \in A} \breve{b}(\{x\}, \phi)] \cup [\cup_{y \in B} \breve{b}(\phi, \{y\})]$.

Here $\{1\}$ and $\{3\}$ disconnects $(X, \breve{b}_X)$ and $\{e\}$ and $\{f\}$ disconnects $(Y, \breve{b}_Y)$, but $(X, Y, \breve{b})$ is not $\breve{b}$-disconnected.

Example 6 *Let $X = \{1,2,3,4\}$ and $Y = \{e, f, g\}$. Let $\breve{b}$ be a Binary $\breve{C}$ech Closure operator defined between X and Y as follows.*

$\breve{b}(\{1\}, \phi) = (\{1\}, \{e\})\breve{b}(\{2\}, \phi) = (\{1,2,3\}, \{f\})\breve{b}(\{3\}, \phi) = (\{3,4\}, \{e\})\breve{b}(\{4\}, \phi) = (\{3,4\}, \{e\})\breve{b}(\phi, \{e\})$

For all other $(A, B) \in \wp(X) \times \wp(Y)$,

$\breve{b}(A, B) = [\cup_{x \in A} \breve{b}(\{x\}, \phi)] \cup [\cup_{y \in B} \breve{b}(\phi, \{y\})]$.

Here the binary sets $(\{1\}, \{e\})$ and $(\{4\}, \{g\})$ disconnect the space $(X, Y, \breve{b})$, but $(X, \breve{b}_X)$

and $(Y, \breve{b}_Y)$ are not disconnected $\breve{C}$ech closure spaces

Definition 12 *A binary subset* (A, B) *of a* $B\breve{C}$ *CS,* $(X, Y, \breve{b})$ *satisfies the Hausdorff- Lennes condition in binary case, if*

$$[\breve{b}(A,B) \cap (X \setminus A, Y \setminus B)] \cup [(A,B) \cap \breve{b}(X \setminus A, Y \setminus B)] \neq (\phi, \phi).$$

Lemma 2 *The Hausdorff- Lennes condition is equivalent to the condition*

$$\breve{b}(X \setminus A, Y \setminus B) \cap \breve{b}(A,B) \neq (\phi, \phi).$$

Proof.

$$[\breve{b}(A,B) \cap (X \setminus A, Y \setminus B)] \cup [(A,B) \cap \breve{b}(X \setminus A, Y \setminus B)]$$

$$= \{[\breve{b}(A,B) \cap (X \setminus A, Y \setminus)] \cup \breve{b}(X \setminus A, Y \setminus B)\} \cap \{[\breve{b}(A,B) \cap (X \setminus A, Y \setminus B)] \cup (A,B)\}$$

$$= [\breve{b}(A,B) \cup \breve{b}(X \setminus A, Y \setminus B)] \cap [(X \setminus A, Y \setminus B) \cup \breve{b}(X \setminus A, Y \setminus B)]$$

$$\cap [\breve{b}(A,B) \cup (A,B)] \cap [(X \setminus A, Y \setminus B) \cup (A,B)]$$

$$= (X,Y) \cap \breve{b}(X \setminus A, Y \setminus B) \cap \breve{b}(A,B) \cap (X,Y), \text{since}(\mathrm{A,B}) \subseteq \breve{\mathrm{b}}(\mathrm{A,B}), \forall(\mathrm{A,B}) \subseteq (\mathrm{X,Y})$$

$$= \breve{b}(X \setminus A, Y \setminus B) \cap \breve{b}(A,B)$$

Proposition 12 *If a* $B\breve{C}$ *CS,* $(X, Y, \breve{b})$ *is connected then each of its proper binary subsets* (A, B) *satisfies the Hausdorff- Lennes condition in binary case.*

Proof. Let $(A,B) \subset (X,Y)$ doesnot satisfy the Hausdorff- Lennes condition in binary case.

Then $\breve{b}(A,B) \cap \breve{b}(X \setminus A, Y \setminus B) = (\phi, \phi)$. $(A,B) \cup (X \setminus A, Y \setminus B) = (X,Y)$.

Also $(A,B) \subseteq \breve{b}(A,B), \forall (A,B) \subset (X,Y) \Rightarrow \breve{b}(A,B) \cup \breve{b}(X \setminus A, Y \setminus B) = (X,Y)$.

So taking $(A_1, B_1) = (A,B) and (A_2, B_2) = (X \setminus A, Y \setminus B)$, disconnects $(X, Y, \breve{b})$.

Definition 13 *A binary* $\breve{C}$ *ech closure space,* $(X, Y, \breve{b})$ *is said to be feebly disconnected if there exists two nonempty binary sets* (A_1, B_1) *and* (A_2, B_2) *such that* $(A_1, B_1) \cup \breve{b}(A_2, B_2) = (X,Y) = \breve{b}(A_1, B_1) \cup (A_2, B_2)$ *and* $(A_1, B_1) \cap \breve{b}(A_2, B_2) = (\phi, \phi) = \breve{b}(A_1, B_1) \cap (A_2, B_2)$.

Remark 6 $(X, Y, \breve{b})$ *is feebly disconnected neednot imply it is disconnected. Also* $(X, Y, \breve{b})$ *is disconnected neednot imply it is feebly disconnected, as seen in the following examples.*

Example 7 *Let* $X = \{1,2,3,4\}, Y = \{e, f, g\}$.

Let $\breve{b} : \wp(X) \times \wp(Y) \to \wp(X) \times \wp(Y)$ be defined as follows:

$\breve{b}(\{1\}, \phi) = (\{1,2\}, \{g\})$, $\breve{b}(\{2\}, \phi) = (\{2,3\}, \{g\}), \breve{b}(\{3\}, \phi) = (\{3,4\}, \{g\})$, $\breve{b}(\{4\}, \phi) = (\{3,4\},\{$

For all other $(A,B) \in \wp(X) \times \wp(Y)$,

$\breve{b}(A,B) = [\cup_{x \in A} \breve{b}(\{x\}, \phi)] \cup [\cup_{y \in B} \breve{b}(\phi, \{y\})]$.

Here the binary sets $(\{1\}, \{e\})$ and $(\{4\}, \{f\})$ feebly disconnect $(X, Y, \breve{b})$, since

$(\{1\}, \{e\}) \cup \breve{b}(\{4\}, \{f\}) = (X,Y) = \breve{b}(\{1\}, \{e\}) \cup (\{4\}, \{f\})$ and

$(\{1\},\{e\}) \cap \breve{b}(\{4\},\{f\}) = (\phi,\phi) = \breve{b}(\{1\},\{e\}) \cap (\{4\},\{f\})$.

But $(X,Y,\breve{b})$ is not disconnected.

Example 8 *Let* $X = \{1,2,3,4\}, Y = \{e, f, g\}$.

Let $\breve{b} : \wp(X) \times \wp(Y) \to \wp(X) \times \wp(Y)$ be defined as follows:

$\breve{b}(\{1\},\phi) = (\{1,2\},\phi)$, $\breve{b}(\{2\},\phi) = (\{2\},\phi), \breve{b}(\{3\},\phi) = (\{3\},\phi)$, $\breve{b}(\{4\},\phi) = (\{3,4\},\phi), \breve{b}(\phi,\{e\},$

For all other $(A,B) \in \wp(X) \times \wp(Y)$,

$\breve{b}(A,B) = [\cup_{x \in A} \breve{b}(\{x\},\phi)] \cup [\cup_{y \in B} \breve{b}(\phi,\{y\})]$.

Here the binary sets $(\{1\},\{e\})$ and $(\{4\},\{f\})$ disconnect $(X,Y,\breve{b})$, since

$\breve{b}(\{1\},\{e\}) \cup \breve{b}(\{4\},\{f\}) = (X,Y)$ and

$\breve{b}(\{1\},\{e\}) \cap \breve{b}(\{4\},\{f\}) = (\phi,\phi)$.

But $(X,Y,\breve{b})$ is not feebly disconnected.

5 Compactness

Definition 14 *Let* $(X,Y,\breve{b})$ *be a* B$\breve{\mathrm{C}}$ *CS. A family* $\{(A_i,B_i) : i \in I\}$ *of binary subsets of* (X,Y) *is a* $\breve{\mathrm{b}}$ *-cover of* $(A,B) \subseteq (X,Y)$ *if* $\{\breve{b}\text{-}int(A_i,B_i) : i \in I\}$ *covers* (A,B), *where* $\breve{b}$ *-* $int(A_i,B_i) = (X \setminus C_i, Y \setminus D_i)$ *and* $(C_i,D_i) = \breve{b}(X \setminus A_i, Y \setminus B_i)$.

A binary set (A,B) in a B$\breve{\mathrm{C}}$ CS, $(X,Y,\breve{b})$*is* $\breve{\mathrm{b}}$-compact if every $\breve{\mathrm{b}}$-cover of (A,B) has a finite subcover. If (X,Y) is a $\breve{\mathrm{b}}$-compact subset of itself, we say that $(X,Y,\breve{b})$ is $\breve{\mathrm{b}}$-compact.

Proposition 13 *If a* B$\breve{\mathrm{C}}$ *CS,* $(X,Y,\breve{b})$ *is* $\breve{\mathrm{b}}$*-compact then each binary subset of* (X,Y) *is* $\breve{\mathrm{b}}$*-compact.*

Proof. Proof follows directly from the definitions.

Proposition 14 *A* B$\breve{\mathrm{C}}$ *CS,* $(X,Y,\breve{b})$ *is* $\breve{\mathrm{b}}$*-compact if given any family,* F *of binary subsets of* (X,Y) *with the finite intersection property,* $\cap\{\breve{b}(E,F) : (E,F) \in \mathsf{F}\} \neq (\phi,\phi)$.

Proof. Assume that $(X,Y,\breve{b})$ is a B$\breve{\mathrm{C}}$ CS with the given condition. Let $\mathsf{F} = \{(E_i,F_i) : i \in \mathsf{I}\}$ be a $\breve{\mathrm{b}}$-cover of (X,Y). Then by the definition of $\breve{\mathrm{b}}$-cover, the family $\{\breve{b}(X \setminus E_i, Y \setminus F_i)\}$ has an empty intersection. Therefore there exists a finite subfamily $\{\breve{b}(X \setminus E_i, Y \setminus F_i) : i \in \mathsf{J}\}$, which has an empty intersection. This means that $\{(E_i,F_i) : i \in \mathsf{J}\}$ is a finite $\breve{\mathrm{b}}$-subcover.

Definition 15 *A binary set* (A,B) *in a* B$\breve{\mathrm{C}}$ *CS,* $(X,Y,\breve{b})$ *is countably* $\breve{\mathrm{b}}$*-compact if every* $\breve{\mathrm{b}}$*-cover of* (A,B) *has a countable subcover. If* (X,Y) *is a countably* $\breve{\mathrm{b}}$*-compact subset of itself, we say that* $(X,Y,\breve{b})$ *is countably* $\breve{\mathrm{b}}$*-compact.*

Proposition 15 *If a* B$\breve{\mathrm{C}}$ *CS,* $(X,Y,\breve{b})$ *is countably* $\breve{\mathrm{b}}$*-compact then each binary subset of* (X,Y) *is countably* $\breve{\mathrm{b}}$*-compact.*

Proof. Proof follows directly from the definitions.

Definition 16 *A binary set* (A,B) *in a* $B\breve{C}CS$, $(X,Y,\breve{b})$ *is said to be* $\sigma\ \breve{b}$*-compact if* (X,Y) *can be written as the union of countably many* $\breve{b}$*-compact binary subsets.*

Proposition 16 *A binary set* (A,B) *in a* $B\breve{C}CS$, $(X,Y,\breve{b})$ *is* $\breve{b}$*-compact* $\Rightarrow$ *it is countably* $\breve{b}$*-compact* $\Rightarrow$ *it is* $\sigma\ \breve{b}$*-compact.*

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