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Orbit-Counting theorem in NG Groups not Subset of S_3

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Abstract: In this paper, we rewrite orbit-counting theorem in NG-groups. Some other authors called it Burnside's counting theorem, the Cauchy Frobenius lemma. It is a result in group theory which is often useful in taking account of symmetry when counting mathematical objects.)
Keywords: Finite groups, Symmetric groups, fixed points.

I. INTRODUCTION

In groups theory [1],[2], the set of all isomorphisms from a set A itself is a group under composition of morphisms, we denoted it by Autζ(A) for any object A in a category ζ . The symmetric group on a set A is a group that consists of all invertible maps from A to itself, we denote by Sn. The order |Sn| of the symmetric group Sn is n!. The NG-group which consisting of transformations on a non-empty set A and the group has no bijection as its element. It is well known that any NG group on a set A with cardinality n has an order not greater than n-1!. We rewrite the orbit-counting theorem in NG-groups. It is a result in group theory which is often useful in taking account of symmetry when counting mathematical objects.

Definition 1.1 The transformation mapping f on X, we called an element x a fixed point if $f(x) = x$, otherwise it's a moved point.

Definition 1.2. The NG-groups is groups which consisting of transformations on a non-empty set A and the group has no bijection as its elements.

Note that for $f\in$NG ,$fix(g)$ is the number of $a\in A$ such that $ag = a$i.e. $fix(a) =a$ the number of fixed points of $a\in A$.

Example 1.1. Consider the set A= {1, 2, 3}, there are 27 transformations mappings from *A to A*. We know the symmetric groups of A is S_3 = {(1, 2, 3),(2, 3, 1),(3, 1, 2),(1, 3, 2),(3, 2, 1),(2, 1, 3)}.So, *S3* is a group. But, it is not an abelian group.Moreover, there exists some groups that are subsets of *Trans(a)*, but not subsets of the *Sym(a)*. We know, every group must contain the identity element. From this point, we will choose an element of *Trans(a)* to be identity element. We repeat this method for all the elements of*Trans(a)*. In addition, we will find all groups that cannot be subset of the symmetric group S_3. For all elements of *Trans(a)*, we introduce the elements of *Trans(a) as*: {(1, 1, 1),(1, 1, 2),(1, 1, 3),(1, 2, 1),(1, 2, 2),(1, 2, 3),(1, 3, 1),(1, 3, 2),(1, 3, 3), (2, 1, 1),(2, 1, 2),(2, 1, 3),(2, 2, 1),(2, 2, 2),(2, 2, 3),(2, 3, 1),(2, 3, 2),(2, 3, 3),(3, 1, 1),(3, 1, 2),(3, 1, 3),(3, 2 , 1),(3, 2, 2),(3, 2, 3),(3, 3, 1),(3, 3, 2),(3, 3, 3)}.The groups of order 2 are: NG1 = {(1, 1, 3),(3, 3, 1)}, NG2 = {(1, 2, 1),(2, 1, 2)},NG3 = {(1, 2, 2),(2, 1, 1)}, NG4 = {(1, 2, 3),(1, 3, 2)}, NG5 = {(1, 2, 3),(2, 1, 3)},NG6 = {(1, 2, 3),(3, 2, 1)}, NG7 = {(1, 3, 3),(3, 1, 1)}, NG8 = {(2, 2, 3),(3, 3, 2)},and NG9 = {(2, 3, 2),(3, 2, 3)}. But, NG4 = {(1, 2, 3),(1, 3, 2)}, NG5 = {(1, 2, 3),(2, 1, 3)}, and NG6 = {(1, 2, 3), (3, 2, 1)} are subsets of *Sym(A)*.

The only groups of mapping on a set A with respect to function compositions which are not subsets of symmetric groups are: NG1 = {(1, 1, 3),(3, 3, 1)}, NG2 = {(1, 2, 1),(2, 1, 2)}, NG3 = {(1, 2, 2),(2, 1, 1)}, NG7 = {(1, 3, 3),(3, 1, 1)}, NG8 = {(2, 2, 3),(3, 3, 2)}, and NG9 = {(2, 3, 2),(3, 2, 3)}. And, the groups of order 3 are: {(1, 2, 3),(2, 3, 1),(3, 1, 2)} which is subset of Sym(A). Moreover, there are no groups of order greater than 3 that cannot be subset of Sym(A).So, the only groups of mapping on a set A with respect to function compositions that cannot be subsets of symmetric groups are six groups of order 2;
NG1 = {(1, 1, 3),(3, 3, 1)}, NG2 = {(1, 2, 1),(2, 1, 2)}, NG3 = {(1, 2, 2),(2, 1, 1)},
NG7 = {(1, 3, 3),(3, 1, 1)}, NG8 = {(2, 2, 3),(3, 3, 2)}, NG9 = {(2, 3, 2),(3, 2, 3)}.
We can see, if $A = \{a, b, c\}$, $NG = \{e, f\} \subset AA$, *where* $e(a) = a, e(b) = a, e(c) = c$,
$f(a) = c, f(b) = c, f(c) = a$, *we get:*$NG1 = \{e(a, b, c) = (a, a, c), f(a, b, c) = (c, c, a)\}$;

where e(a) = a, e(b) = c, e(c) = c, f(a) = c, f(b) = a, f(c) = a. We get:NG7 = {e(a, b, c) = (a, c, c), f(a, b, c) = (c, a, a)}; where e(a) = a, e(b) = b, e(c) = b, f(a) = b, f(b) = b, f(c) = a, we get:NG3 ={e(a, b, c) = (a, b, b), f(a, b, c) = (b, a, a)};
where e(a) = b, e(b) = b, e(c) = c, f(a) = c, f(b) = c, f(c) = b, we getNG8 = {e(a, b, c) = (b, b, c), g(a, b, c) = (c, c, b)}; where e(a) = a, e(b) = b, e(c) = a, f(a) = b, f(b) = a, f(c) = b, we get:NG2 = {e(a, b, c) = (a, b, a), f(a, b, c) = (b, a, b)};where e(a) = b, e(b) = c, e(c) = b, f(a) = c, f(b) = b, f(c) = c, we get:
NG9 = {e(a, b, c) = (b, c, b), f(a, b, c) = (c, b, c)}.So, if A = {a, b, c}, *NG = {e, f}* $\subset A^A$, *we see* $e^2 = e$, *ef = fe = f,* $f^2 = e$.

Example 1.2. Suppose *A* = {1, 2, 3}. Consider the transformation mapping on *A*.*f* = (1,1,3) It has two fixed points {1, 3} since *f*(1) = 1 and *f*(3) = 3 and *f*(2) ≠ 2 = 1, then 2 ∈*A* is movement point.We defined the number of fixed points in NG as | *fix(f)* |= 2 and | *Move(f)* |= 1.i.e. we have two fixed points and one movement point.

We have one movement point, if m is movement point, we get NG = {*e, f*} where*e* = (1, 2, *m*), where *m* = 1or 2. If *m*= 1, we get NG2, if m = 2, we get NG3.And, *f* not has fixed point. Also,3 not loop. Where e = (1, *m*, 3) , then m can be 1 or 3. If *m* = 1, we get NG1 and if m = 3, we get NG7. An addition, f does not have fixed points. Also,2 not loop. If e = (m, 2, 3), and m can be 2 or 3. If m = 2, we get NG8 .And if m = 3, then we get NG6. Moreover, f not have fixed points. Also,1 not loop.

Now, we can generalization our result for a finite set A. For a finite set A, the identity could be e = (1, 2, ..., n − 1, m). We have n−1 possibility of m, where m = 1, 2, 3, ..., or n−1, So, we can get the numbers of maximum groups, NGMax, of G that are mapping on a set A with respect to function compositions which are not subsets of symmetric groups by using the fixed points. The number of maximum of NG has n ≥ 3 elements by NGMax=(n−1) × (n!(n−1)×!(1)!)
it means NGMax = (n−1)×n.
So, we can introduce the easy proposition

Proposition 1.1. The maximal of maximum of NG has n ≥ 3 elements is (n−1)×n.
i.e. NGMax=(n−1)×n.

Theorem 1.1. Suppose NG a group which consists of non-bijective transformations on a non-empty set A. For any fixed point i∈ A,

| NGi|×| iNG |=|NG|.

Remark 1.1. The Theorem 1.1 is work for only if a ∈ A is fixed point.
Proposition 1.2. The maximal of ofNGMax has n ≥ 3 elements is (n−1)×n.
i.e. NGMax= (n−1)×n.
Proof. It easy to prove it by mathematical induction.

Proposition 1-3. For every NG-groups of order 2, G={f , g},on a finite set X, n≥3, , the difference of number of fixed points of the elements of NG is 2. i.e|| ffix| -| gfix|| =2

Proof : We know that NG is a groups, then NG={f,g} , such that (| ffix| = n-i , | gfix| =(n-2)-i) where 1≤i≤n-2. We prove it by Mathematical induction;
Frist, we check for n=3; Suppose X={1,2,3}, | X| =3; So, NG={f,g} , such that (| ffix| =3-i , | gfix| =(3-2)-i) where 1≤i≤3-2=1. It is true for n=3, f has 2 fixed point. But, g has 0 fixed points. Second, we assume that it is true for n=k; So, we have NG-groups (NG={f,g}) , such that (| ffix| =k-i , | gfix| =(k-2)-i) where 1≤i≤k-2;
Third, we try to prove it is true for n=k+1 ; Suppose, | X| =k+1; So, (| ffix| =(k+1)-i , | gfix| =((k+1)-2)-i) where 1≤i≤(k+1)-2; That mean, (| ffix| = (k+1)-i , | gfix| =((k-1)-i) where 1≤i≤(k-2)+1; We know, it is true for 1≤i≤k-2; we just check for n= k+1; So, || ffix| - | gfix|| = (k+1)–i - ((k-2)+1-i); = k+1–i -k+2-1-i =2; so, it is true for n=k+1.
Then, from Mathematical induction, it is true for n∈N and n≥3.

II. OUR RESULT

We will rewrite the definition of the stabilizer of a point as an algebraic concept: it is the setof group elements which fix the point.
Definition 2.1. Let NG be a group which consists of non-bijective transformations on anon-empty set A. Suppose a be an element of A. Then *Satb(a) = NGa = {f ∈NG : f.a=a}*is called the stabilizer of a and consists of all the transformation NG that produce groupfixed points in *A*.i.e., that send a to itself.

Suppose that NG is a group which consists of non-bijective transformations on a non-empty set A. If a∈A, then the point-stabilizer NGa is the subgroup of NG formed by the elements which fix(a). Moreover, we will write the definition of orbit of a point as a geometric concept: it is the set of places where the point can be moved by the group action and the size of an orbit as its length.

Definition 2.1. Let NG be a group which consists of non-bijective transformations on anon-empty set A. Suppose a be an element of A. Then, the group NG containing a ∈ A isaNG = Orb(a) = {af : f ∈ NG}. And, we called it the orbit of *a*.

Let NG be a group which consists of non-bijective transformations on a non-empty setA. If a∈A, then the orbit of a is the subgroup of A formed by the elements which fix(a).

Example 2.1. Let A = {1, 2, 3} and NG= {(1, 1, 3),(3, 3, 1)}. So, the fixed points are{1, 3}. The group NG containing 1 ∈ A is 1NG = {if : f ∈ NG} = 1NG = {1, 3} = 2NG = 3NG.The Stabilizer in a group NG of i is NGi = {f ∈NG : if = i}.NG1 = {(1, 1, 3)}. Note that: | NG1 | × | 1NG |= 1 × 2 = 2 =|NG|.

Remark 2.1. Note that, this is work for only if a ∈ A is fixed point. For example, in previous example if we take i = 2, then NG2 = φ. So, | NG2|×|2NG |= 0 × 2 = 0 ≠| NG |.

Proposition 2.1. Let NG be group that cannot subsets of S3. For each f∈NG, let fix(A) the set of elements in A that are fixed by f. Orbit-Counting theorem asserts the following formula for the number of orbits, denoted | A/NG |, $|\boldsymbol{A/NG}| = \frac{\mathbf{1}}{|\boldsymbol{NG}|}\sum_{\boldsymbol{f\in NG}}|\boldsymbol{fix}(\boldsymbol{A})|$ i.e$|\boldsymbol{orb}(\boldsymbol{i})| = \frac{\mathbf{1}}{|\boldsymbol{NG}|}\sum_{\boldsymbol{f\in NG}}|\boldsymbol{NG_a}|$, Thus the number of orbits is equal to the average number of points fixed by an element of NG.

Proof. First step we re-express the sum over the group elements f∈NG as an equivalent sum over the set elements a ∈A :$\sum_{\boldsymbol{f\in NG}}|\boldsymbol{fix}(\boldsymbol{A})|$=|{(f,a)∈NG×A |f.a=a|=$\sum_{\boldsymbol{a\in A}}|\boldsymbol{NG_a}|$,By orbit-stabilizer theorem we have a natural bijection for each a in A between the orbit of a.NG.a = f.a.f∈NG⊆ A, and the set of left cosets NG/NGa of its stabilizer subgroup NGa. With Lagrange's theorem this implies |NG.a | = [NG: NGa] = |NG |/|NGa|. So, the sum over the set A can be rewritten as$\sum_{\boldsymbol{a\in A}}|\boldsymbol{NGa}|$=$\sum_{\boldsymbol{a\in A}}\left|\frac{\boldsymbol{NG}}{\boldsymbol{NG.a}}\right| = |\boldsymbol{NG}|\sum_{\boldsymbol{a\in A}}\left|\frac{\mathbf{1}}{\boldsymbol{NG.a}}\right|$.

Finally, the sum over A may be broken up into separate sums over each individual orbit.since A is the disjoint union of all its orbits in A/NG:

$$\sum_{\boldsymbol{a\in A}}\left|\frac{\mathbf{1}}{\boldsymbol{NG.a}}\right| = \sum_{\boldsymbol{B\in A/NG}}\sum_{\boldsymbol{a\in B}}\left|\frac{\mathbf{1}}{\boldsymbol{B}}\right| = \sum_{\boldsymbol{B\in A/NG}}\mathbf{1} = |\boldsymbol{A/NG}|.$$

Putting everything together gives the desired result:

$$\sum_{\boldsymbol{f\in NG}}|\boldsymbol{fix}(\boldsymbol{A})| = |\boldsymbol{NG}|.|\boldsymbol{A/NG}|$$

This proof is essentially also the proof of the class equation formula, simply by takingthe action of NG on itself (A =NG) to be by conjugation, f.$a = faf^{1}$;in which case NGa instantiates to the centralizer of a in NG.

Example 2.2. Let A = {1, 2, 3} and NG = {(1, 1, 3),(3, 3, 1)}. So, the fixed points are {1, 3}. The group NG containing 1 ∈ A is Orb(1) =1NG = {1, 3} = 2NG = 3NG.The Stabilizer in a group NG of i is NGi = {f ∈NG : if = i}. |Orb(i)|= 1.NG1 = {(1, 1, 3)},NG3= {(1, 1, 3)}.So, |NG1|= |NG3|= 1.|A/NG|=$\frac{\mathbf{1}}{|\boldsymbol{NG}|}\sum_{\boldsymbol{f\in NG}}|\boldsymbol{fix}(\boldsymbol{A})| = \frac{\mathbf{2}}{\mathbf{2}} = \mathbf{1}$= Orb(i),where i is fixed points.

Example 2.3. Let consider NG not subset of S4. Suppose NG ={(1, 2, 2, 4), (4, 2, 2, 1); (1, 4,4,2), (2,4,4,1), (2,1,1,4),(4,1,1,2)}: The fixed points in A are {1, 2,4}; Orb(1)=1NG= {1,2,4}=2NG=3NG = 4NG. |Orb(i)|= 1.NG1 = {(1, 1, 4, 3), (1,4,4, 2)};NG2={(1,2,2,4), (4,2,2,1)};And NG4 ={(1,2,2,4),(1,2,2,4)}.

So,|NG1|= |NG2|=|NG4|=2.|A/NG|=$\frac{\mathbf{1}}{|\boldsymbol{NG}|}\sum_{\boldsymbol{f\in NG}}|\boldsymbol{fix}(\boldsymbol{A})| = \frac{\mathbf{2+2+2}}{\mathbf{6}} = \mathbf{1}$= Orb(i),where i is fixed points.

Proposition 2.2. Every groups on NG -groups on X= {1, 2, 3} has one element which it is has two fixed points.

Proof

For X={1,2,3}, we have six groups that are not subsets of Sym(X) from proposition 1.3 Suppose NG={f,g}, where f and are two mapping from X to X.We should check three cases, if f has zero, one, or three fixed points.

Case one, if f has no fixed point, then g has two fixed points by the proposition 1-3Contradiction;

Case two, if f has one fixed point, then g has three fixed points by the proposition 1.3.Contradiction;

Case three, if f has tree fixed points, which means, f is the identity of Sym(X). Contradiction ;

So, Every groups on Ng -groups on X={1, 2, 3}has one element which it is has two fixed points.

Proposition 2-3. Every NG-group on X={1, 2, 3}, there exits one of mapping has two fixed points and the image of the third point of this mapping will be one of these fixed points. i.e f(c)= a or b.

Proof

Suppose NG={f,g}; we have NG has six groups. And suppose f has two fixed points Since f has two fixed points, then we have three cases for f ;

if f(a,b,c)= (a,b,*),then f(c)≠ c. So, f(c)= a or b. Then, we have two groups

if f(a,b,c)= (a,*,c), then f(b)≠ b, So, f(b)=a or c. Then We have two groups

if f(a,b,c)= (*,b,c), then f(a)≠ a, So, f(a)=b or c. Then, we have two groups

From these three cases, we have six NG-groups.

Proposition 2-4 Every NG-groups on a finite X has one element having one fixed point.

Proof

We assume there exits NG-groups have one element having no fixed points. If we consider the NG-groups of order 2, we have the difference between of number of fixed points of the elements of NG is 2. So, it is contradiction with our assumption.

Now, we are interesting in find the answer of this question ;

Q; If the difference of number of fixed points of elements of G is not 2, is there exits NS-groups? And what is the order of G?

Proposition 2-5. Let X be a finite Set with ∣X∣ ≥3, the number of NG-groups of order 2 is∣ X∣ * (Max∣ ffix∣)(max∣ ffix∣ -1).

Proof :

Suppose X={1,2,3,…}, We will prove it by Mathematical induction;Frist, we check for n=3; Suppose X={1,2,3}, ∣X∣ =3; So, NG={f,g} , such that Max∣ffix∣ = 2,So, n * (Max∣ ffix∣)(Max∣ ffix∣ -1) = 3 * (2)(2-1) =3*2=6. It is true for n=3..n=4, So, n* (Max∣ ffix∣)(Max∣ ffix∣ -1) = 4*(3)(3-1) =4*9=36.

Second, we assume it is true for n=k; So, we have Max∣ ffix∣ = k-1, k*(k-1)(k-1) =k*((k)(k-1) –1) = kk-k;

Third, we try to prove it is true for n=k+1 ;

Suppose, ∣X∣ =k+1; So, max∣ffix∣ = k, (k+1)*(k)(k-1)=k*kk-1+(k)(k-1)= k+(k)(k-1) it is true by assumption; That mean it is true for all n∈N.

Definition 2.2. Let G be a NG-group on a set X and x be an element of X. Then

$$G_x = \{g \in G : g(x) = x\}$$

is called the stabilizer of x and consists of all the transformation G that produce group fixed points in x, i.e., that send x to itself.

Examples 2.4. Let X ={1, 2, 3, 4} and NG= { (1,1,4,3) (1,1,3,4) (4,4,1,3) (4,4,3,1) (3,3,1,4) (3,3,4,1)}; The NG-group containing i∈X is 1G= {if :f ∈ G}.

1G = {1, 3, 4} = 2G =3G =4G.

Stabiliser in NG of i is NGi = {f∈ NG| if = i}.NG1= {(1, 1, 4, 3), (1, 1, 3, 4)}.

Notice: |NG1| · |1G| = 3 × 2 = 6 = |NG|.

Proposition 2.6. For any NG-group and any fixed point i∈ X, |NGi|·|iNG| = |NG|.

Proof

Write iNG = {i1, . . . ,ir} and K = NGi = {k1, . . . , Km}. Choose elements f1 =1, f2, . . . ,fr such that ifj = ij for each j. Arrange (some of) the group elements like this.

k1g1… k1gr

…………..

…………..

Kmg1…..kmgr

The elements in column j form the cosetkfj . Every element in kgj is maps i to ij .
So this table contains exactly r · s distinct elements of NG.
Claim: every f$\in$ NG is exactly once in
k1f1… k1fr
…………..
…………..
Kmf1…..kmfr

Find ig. By definition of fixed points ig = ij for some j.
Compute h: = ff−1. (Remember: g first, and then f−1.)
Then K :i$\to$i , so, k$\in$K =N G1. Hence f = kgj lies in column j.
Counting:
|NG| = number of elements in the table =r·s=|iNG|*|NGi|.

III. CONCLUSION

We rewrite orbit-counting theorem in NG-groups. Some other authors called it Burnside's counting theorem, the Cauchy Frobenius lemma. It is a result in group theory which is often useful in taking account of symmetry when counting mathematical objects.

REFERENCES

[1]. Isaacs, I. Martin,(2008). Finite Group Theory Graduate Studies in Mathematics, Vol. 92, American,MathematicalSociety Providence, Rhode Island.
[2]. Bogopolski(2008), Intreudction to Group Theory, European Mathematical Society,Printed in Germany.
[3]. Y.L. Wu, X.Y.Wei, (2005), Condition of the groups generated by nonbijective transformations on a set (In Chinese), Journal of Hubei University (Natural Science), 27, 1-3.
[4]. vY.X. Zhu, J.K.Hai((2003), Groups Generated by Non -bijective Transformations on a Set and Their Representations in the Matrix Algebra (In Chinese), Journal of Capital Normal University (Natural Science), 24, 5-14.
[5]. E. James. Baumgartner, SaharonShelah,and Simon Thomas, Maximal Subgroups of Infinite Symmetric Groups, Notre Dame Journal of Formal Logic, Volume 34, Number 1, Winter 1993.
[6]. J. East, J. D. Mitchell, and Y. P´eresse, Maximal subsemigroups of the semigroup of all mappings on an infinite set, 12 Feb 2013.
[7]. Ralph W.Ball, Maximal subgroups of the symmetric group,AMS, 1965.
[8]. Jason Fulman, and Robert Guralnick, On fixed points of permutationspersi diaconis,2007
[9]. Cameron. P. J and A.M. Cohen, On the number of fixed points free elements in a permutations groups, Discrete Mathematics 106/107(1992) 135-138.