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A Meshfree Method for Coupled Burgers' Equations

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Abstract: Coupled with the finite difference method, the radial basis function collocation method can be extended to solve the time-dependent problems. This paper presents a meshfree scheme for the direct approximation of the 2D coupled Burgers' equations. This can be done by using the two different schemes in the sense of radial and non-radial basis functions. The time variable in the first radial scheme is treated equally to the normal space variables to construct a 'isotropic' space-time radial basis function. The non-radial scheme constructed a sensible relationship between space variables and time variable. An advantage of this space-time meshfree scheme is that the solution of both time-dependent and time independent equations can be combined in a unified way. Numerical results show that the proposed semi-analytical meshfree schemes are accurate, easy-to-programm and efficient for the coupled Burgers' equations.
Keywords: Radial basis functions; coupled Burgers' equations; meshless methods.

1 Introduction

A large number of mathematical models in mathematical physics can be described by the Klein-Gordon equations. It has attracted much attention in studying the recurrence of initial states, solitons and condensed matter physics [1, 2].

For such time-dependent problems, it is very difficult to get the corresponding theoretical/analytical solutions. Thus one should consider numerical approximations to the Klein-Gordon equations. A variety of numerical methods have been proposed and compared for solving the Klein-Gordon equations [3]. Almost all these numerical methods are based on the other methods [4, 5].

To avoid the mesh generation, the radial-basis-function-based meshfree methods have fascinated many scholars' attention [6, 7, 8, 9, 10, 11]. It should be pointed that the above-mentioned numerical method are all two-step methods, i.e., the finite difference method is used to discretize the time varaible and then another method can be used to find numerical solutions for time-independent problems.

In this paper, we propose a space-time semi-analytical meshfree method, which is a one-step method, for the one-dimensional Klein-Gordon equations and two-dimensional coupled Burgers' equations. For the time-dependent Klein-Gordon equations, two different schemes are proposed for the basis functions from radial and non-radial aspects. The time variable in the first radial scheme is treated equally as space variables which yields a 'isotropic' space-time radial basis function. A realistic relationship between space variables and time variable is investigated by the non-radial scheme. Under such circumstances, the time variable and space variables can be treated simultaneously during the whole solution process and the Klein-Gordon equations can be solved in a direct way.

2 Problem Description

In this paper, we consider the general mathematical formulation of 2D coupled Burgers' equations

$$\frac{\partial U}{\partial t}+U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}-\frac{1}{Re}\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)=0,\ (x,y)\in\Omega \tag{1}$$

$$\frac{\partial V}{\partial t}+U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}-\frac{1}{Re}\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right)=0,\ (x,y)\in\Omega \tag{2}$$

with initial and boundary conditions:

$$U(x,y,0)=g_1(x,y), V(x,y,0)=g_2(x,y),\ (x,y)\in\Omega \tag{3}$$

$$U(x,y,t)=g_3(x,y,t), V(x,y,0)=g_4(x,y,t),\ (x,y)\in\partial\Omega \tag{4}$$

where Re is a real constant known as the Reynolds number.

3 Formulation of the space-time radial and non-radial basis functions

For traditional numerical techniques, Eqs. (1)-(3) can be solved by using the two-level finite difference approximations or integral transform methods. In order to overcome the two-level strategy, we propose direct meshless methods by using space-time radial and non-radial basis functions.

As is known to all, the radial basis functions (RBFs) are 'isotropic' for Euclidean spaces. For steady-state problems, the approximate solution can be written as a linear combination of RBFs with 2D or more higher dimensions. Take the famous Multiquadric (MQ) RBF as an example

$$\phi_{MQ}(r_j) = \sqrt{1+(\varepsilon r_j)^2} \tag{5}$$

where $r_j = \|X - X_j\|$ is the Euclidean distance between two points $X=(x,y)$ and $X_j=(x_j,y_j)$, ε is the RBF shape parameter.

However, there is only one space variable x for the one-dimensional Klein-Gordon equation, the traditional RBFs are unapplicable in the direct sense. For this reason, we propose a simple meshless method by combining the space variable x and time variable t from the perspective of radial and non-radial. More specifically, the interval $[a,b]$ is evenly divided into segments firstly $a = x_0 < x_1 < \ldots < x_n = b$ with corresponding finess $h=(b-a)/n$. The time variable is evenly chosen from the initial time $t_0 = 0$ to a final time $t_n = T$ as $0 = t_0 < t_1 < \ldots < t_n = T$ with time-step $\Delta t = T/n$. Then the space-time radial basis function can be constructed as

$$\varphi_{MQ}(r_j) = \sqrt{1+c^2 r_j^2} \tag{6}$$

$r_j = \|P - P_j\|$ is the Euclidean distance between two points two points $P=(x,t)$ and $P_j=(x_j,t_j)$. Besides, we can construct the space-time non-radial basis function which has the following expressions

$$\varphi_{NMQ}(P,P_j) = \sqrt{1+(x-x_j)^2 + c^2(t-t_j)^2} \tag{7}$$

where c is a parameter which reflects a realistic relationship between space variable x and time variable t.

We note that the space-time non-radial basis function which is product of two positive definite functions on space dimension and time dimension is investigated in [?, ?]. For the MQ case, one have

$$\varphi'_{NMQ}(P,P_j) = \sqrt{1+c^2(x-x_j)^2}\,\sqrt{1+c^2(t-t_j)^2}. \tag{8}$$

However, the corresponding numerical results is not well in dealing with the problems in this research.

For two-dimensional cases, the space-time radial and non-radial basis functions can be easily obtained

$$\varphi_{MQ}(r_j) = \sqrt{1+c^2 r_j^2} \tag{9}$$

$$\varphi_{NMQ}(P,P_j) = \sqrt{1+(x-x_j)^2 + (y-y_j)^2 + c^2(t-t_j)^2} \tag{10}$$

with $r_j = \|P - P_j\|$ is the Euclidean distance between two points two points $P=(x,y,t)$ and $P_j = (x_j, y_j, t_j)$.

4 Implementation of the space-time simi-anlaytical meshfree method (SSMM)

Here, we consider the initial boundary value problem Eqs. (1)-(3) to illustrate the direct meshless method (SSMM). Based on the definition of space-time radial and non-radial basis functions, Eqs. (1)-(3) can be solved directly in a one level approximation. The approximate solution of the function $u(x,t)$ has the form

$$\bar{u}(\cdot) \approx \sum_{j=1}^{N} \lambda_j \varphi_j(\cdot). \tag{11}$$

with $\{\lambda_j\}_{j=1}^n$ the unknown coefficients.

To illustrate the space-time simi-anlaytical meshfree method, we choose collocation points on the whole physical domain which include N_I internal points $\{P_i = (x_i,t_i)\}_{i=1}^{N_I}$, N_t initial boundary points

$\{P_i = (x_i, t_i)\}_{i=N_I+1}^{N_I+N_t}$ and N_b boundary points $\{P_i = (x_i, t_i)\}_{i=N_I+N_t+1}^{N}$. Based on the traditional collocation approach, by substituting Eq. (11) into Eqs. (1)-(3), we have the $N \times N$ linear algebraic system

$$\mathbf{AX} = \mathbf{f}, \tag{12}$$

where A is $N \times N$ known matrix, X is $N \times 1$ vectors, $\mathbf{f}$ is $N \times 1$ vectors. This can be solved by the backslash computation in MATLAB codes. From the above procedures, we can find that the implementation of the proposed space-time simi-anlaytical meshfree method is very simple.

5 Numerical experiments

To compare with the previous literatures, we consider using the maximum error, absolute error and root mean square error (RMSE) [12, 13] defined as below:

$$\text{RMSE} = \sqrt{\frac{1}{N_t}\sum_{k=1}^{N_t} |u(P_k) - \bar{u}(P_k)|^2}, \tag{13}$$

where $u(\cdot_k)$ is the analytical solution at test points $\{P_k\}_{k=1}^{N_t}$ and $\bar{u}(\cdot_k)$ is the numerical solutions at the test points $\{P_k\}_{k=1}^{N_t}$. N_t is the number of test points on the physical domain. The shape parameter $c = 1$ is chosen for the first two one-dimensional Klein-Gordon equations and the shape parameter for the rest two two-dimensional coupled Burgers' equations is chosen by prior numerical results.

For simplicity, we denote the space-time radial basis function Eq. (5) and space-time non-radial basis function Eq. (6) as SSMM1 and SSMM2, respectively.

We consider the 2D Burgers¡¯ equations, with the initial conditions $u(x, y, 0) = x + y$, $v(x, y, 0) = x - y$ and the exact solutions are as follows

$$u(x, y, t) = \frac{x + y - 2xt}{1 - 2t^2}, \tag{14}$$

$$v(x, y, t) = \frac{x - y - 2yt}{1 - 2t^2}. \tag{15}$$

Table 1: Comparison of numerical solutions with the exact solutions for u and v at $t = 0.1$ for Case 4.3.

Test Point	(0.1,0.1)	(0.2,0.2)	(0.3,0.3)	(0.5,0.5)
u-SSMM1	3.86E-05	4.39E-05	1.94E-05	6.95E-07
u [14]	3.31E-06	6.62E-06	9.92E-06	1.65E-05
v-SSMM1	3.55E-05	8.95E-05	2.82E-05	3.74E-06
v [14]	1.05E-06	2.11E-06	3.16E-06	5.27E-06
u-SSMM2	3.05E-05	1.71E-05	9.02E-05	1.57E-06
v-SSMM2	8.69E-06	5.39E-05	9.95E-05	3.79E-06

The computational domain has been taken as $D = \{(x, y) \mid 0 \le x, t \le 0.5\}$. The study compares the presented SSMM with the discrete ADM in [14], the problem is solved at $t = 0.1$ and $t = 0.4$ for uniform mesh $h_x = h_y = 1/8$ and $h_x = h_y = 1/9$, respectively. The uniform mesh $h_x = h_y = 0.025$ used in [14] is smaller than the present SSMM.

The numerical solutions for arbitrary Reynolds numbers are listed in Tables 5 and 6, respectively. From Table 5, we can see that the present SSMM performs better than the discrete ADM at $t = 0.1$ for test point $(x, y) = (0.5, 0.5)$. While the numerical results for the other test points are almost the same. From Table 6, it can be seen that the approximation solutions by SSMM perform better than the discrete ADM at a certain time t for all test points. Therefore it is concluded that the SSMM is an accurate and efficient method to solve a nonlinear system of equations. From practical opinions, the numerical results may reduce with the increase of time t. Numerical results show that the SSMM is more stable than the discrete ADM with the increase of time t.

Table 2: Comparison of numerical solutions with the exact solutions for u and v at $t = 0.4$ for Case 4.3

Test Point	(0.1,0.1)	(0.2,0.2)	(0.3,0.3)	(0.5,0.5)
u-SSMM1	2.37E-06	3.85E-05	8.89E-05	6.61E-06
u [14]	1.02E-04	2.04E-04	3.06E-04	5.10E-04
v-SSMM1	3.73E-06	1.14E-04	1.47E-05	1.81E-04
v [14]	3.55E-04	7.10E-04	1.06E-03	1.77E-03
u-SSMM2	1.99E-05	2.00E-05	4.43E-04	7.22E-06
v-SSMM2	3.26E-05	1.57E-06	1.93E-04	1.51E-06

6 Conclusions

In this paper, a new space-time simi-anlaytical meshfree method is proposed for the one-dimensional Klein-Gordon equations and the two-dimensional coupled Burgers' equations. Two schemes for the basis functions from radial and non-radial aspects. The first scheme is fulfilled by considering time variable as normal space variable to construct a 'isotropic' space-time radial basis function. The other scheme considered a realistic relationship between space variable and time variable which is not radial. Both schemes for the proposed meshless method are simple, accurate, stable, easy-to-programm and efficient for the Klein-Gordon equations. More importantly, the proposed method can be used to nonlinear problems accompanied with iteration methods. The theory of our SSMM procedure can be directly applied to wave propagation, transient heat transfer and thermo-elastic problems with high dimensions.

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