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# Bilinear Optimal Control for Stochastic Wave Equations

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**Abstract:** In this paper, we consider the bilinear optimal control problem for wave equation with multiplicative control and random noise. Optimal control problems for infinite dimensional stochastic equations had been studied by many researchers. Most of the researchers had considered the Hamilton-Jacobi-Bellman(HJB) approach and obtained the optimal feedback laws using associated HJB equation on a Hilbert space. In present work, we do not use HJB approach. Also, Riccati equation framework is not suitable for this problem due to nonlinear nature introduced by multiplicative controls. We first prove the the existence of weak solution of wave equation equation with random noise. Then, we prove the existence of optimal control and give the characterization in the form of optimality system.

**Keywords:** Optimal control; Stochastic wave equation; Random noise; Bilinear control; Brownian motion.

## I. INTRODUCTION

Many physical problems including elasticity problem are modeled by using wave equations. Hence, controllability of wave equations is an important issue in control theory. The controllability of deterministic equations is addressed by many researchers. When we model a physical state, external random noise becomes a serious disturbance. This affects the controllability of the system. Thus it is an important issue in control theory to address the controllability of systems with random noise.

In this paper, we will consider the bilinear optimal control problem for stochastic wave equation with multiplicative controls. Optimal control problems for infinite dimensional stochastic equations had been studied by Barbu and Da Prato[1], Cannarsa and Da Prato[2, 3], Gozzi[4, 5]. In these papers, authors consider the Hamilton-Jacobi-Bellman(HJB) approach and obtain the optimal feedback law using associated HJB equation on a Hilbert space. In most of the papers backward stochastic differential equations (BSDEs) are applied to solve stochastic optimal control problems. For example, see [6, 7, 8] and references therein. The maximum principle had been used in some of these papers to find necessary/sufficient conditions for optimality of stochastic partial differential(SPDEs). In present work, we do not use HJB approach. Also, Riccati equation framework is not suitable for this problem due to nonlinear nature introduced by multiplicative control. The paper is organized as follows. In section 2, we state preliminaries and formulate the optimal control problem. We will prove the existence and uniqueness of weak solution to the stochastic wave equation in section 3. Existence of an optimal control is proved in section 4. Also, the characterization of an optimal control through the optimality system is given.

## II. PRELIMINARIES

We consider the linear stochastic wave equation defined by

$$\begin{aligned} &y_{tt} = \Delta y + uy + f + \sum_{j=1}^{N} (g_j + f_j y)\frac{dW_j}{dt}, && \text{in } D \times (0,T),\\ &y(x,t) = 0, && \text{on } \partial D \times [0,T],\\ &(y(x,0), y_t(x,0)) = \phi_0(x), && x \in D, \end{aligned} \tag{1}$$

where $D$ is a bounded subset in $\mathbb{R}^n$ with smooth boundary $\partial D$. $\{W_j(t)\}_{j=1}^N$ is a set of mutually independent standard Brownian motions over a given stochastic basis $\{\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}\}$, where $\{\mathcal{F}_t\}$ is a filtration over the $\sigma$-algebra $\mathcal{F}$ and $\mathbb{P}$ is probability measure on $\Omega$. The multiplicative control $u$ is a bounded random function such that $u(t)$ is an $L^\infty(D)$-valued predictable process over $\{\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}\}$, and $u \in L^\infty(\Omega, \mathcal{F}_t; L^\infty(D \times (0,T)))$. If $X$ is a Banach space and $\Sigma$ is a $\sigma$-algebra over $\Omega$, then $L^2(\Omega, \Sigma; X)$ denotes the set of all $X$-valued $\Sigma$-measurable functions $f$ with $\mathbb{E}(\| f \|_X^2) < \infty$. If we have $\Sigma = \mathcal{F}$, we simply write it as $L^2(\Omega; X)$. For separable Hilbert space $X$, $X$-valued stochastic integrals can be expressed in terms of a complete orthonormal basis for $X$.

For each $j = 1, \cdots, N$, assume that $g_j$ is an $L^2(D)$-valued predictable process and $f_j$ is an $L^\infty(D)$-valued predictable process such that

$$g_j \in L^2(\Omega; L^2(D \times (0,T))) \tag{2}$$

and for some nonnegative constant $b_j$,

$$|f_j(\omega,x,t)| \le b_j, \tag{3}$$

for all $(x,t) \in D \times [0,T]$, for almost all $\omega \in \Omega$.

Let

$$\chi = H_0^1(D) \times L^2(D).$$

**Definition 2.1** A stochastic process $y$ is said to be a weak solution of (1) if

1. $(y(t), y_t(t))$ is $\chi$-valued and $\mathcal{F}_t$ measurable for each $t \in [0,T]$,
2. $(y, y_t) \in L^2(\Omega; C([0,T];\chi))$,
3. $(y(0), y_t(0)) = \phi_0$,

and

$$\begin{aligned}\langle y_t(t), v\rangle = {} & \langle y_t(0), v\rangle - \int_0^t \langle \nabla y(s), \nabla v\rangle ds + \int_0^t \langle u(s)y(s) + f(s), v\rangle ds \\ & + \sum_{j=1}^N \langle g_j(s) + f_j(s)y(s), v\rangle dW_j(s)\end{aligned} \tag{4}$$

holds for all $t \in [0,T]$ and all $v \in H_0^1(D)$, for almost all $\omega \in \Omega$. Here $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2(D)$.

We consider the optimization problem

$$J(u) = \mathbb{E}\left(\frac{1}{2}\int_0^t \int_D |y-z|^2 d\xi dt + \frac{\beta}{2}\int_0^t \int_D |u|^2 d\xi dt\right) \tag{5}$$

constrained by the stochastic wave equation (1). $z$ in (5) is a desired target solution and $\beta$ is a positive constant. Our aim is to minimize the the functional $J$ over the set $\mathcal{U}$ of all admissible controls defined by

$$\begin{aligned}\mathcal{U} = {} & \{u \in L^\infty(\Omega, \mathcal{F}_t; L^\infty(D \times [0,T])) : \exists\ y \in L^2(\Omega; C([0,T]; H_0^1(D))) \\ & \text{corresponding to } u \text{ satisfying (1) and } J(y,u) < \infty\}.\end{aligned}$$

We state the following lemma from [9], which is used to prove the existence and uniqueness of weak solution to the system (1).

**Lemma 2.1 ([9])** For each $j = 1, \cdots, N$, let $\{h_j\}_{j=1}^N$ be a set of random functions such that $h_j$ is an $L^2(D)$-valued predictable process and $h_j \in L^2(\Omega; L^2(D \times (0,T)))$. Let $0 < T < \infty$, $L^2(D)$-valued predictable process $f \in L^2(\Omega; L^2(D \times (0,T)))$, and $\phi_0 \in L^2(\Omega, \mathcal{F}_0; \chi)$ be given. Then there exists a weak solution to the system

$$\begin{aligned} & y_{tt} = \Delta y + f + \sum_{j=1}^N h_j \frac{dW_j}{dt}, && \text{in } D \times (0,T), \\ & y = 0, && \text{on } \partial D \times [0,T] \\ & (y(x,0), y_t(x,0)) = \phi_0(x), && x \in D. \end{aligned} \tag{6}$$

Moreover, this solution is pathwise unique, and

$$\begin{aligned} & \mathbb{E}\left(\sup_{s\in[0,t]}\left(\| y_s(s) \|_{L^2(D)}^2 + \| \nabla y(s) \|_{L^2(D)}^2\right)\right) \\ & \le C\mathbb{E}\left(\int_0^t \| f(s) \|_{L^2(D)} ds\right) + C\mathbb{E}\left(\| y_t(0) \|_{L^2(D)}^2 + \| \nabla y(0) \|_{L^2(D)}^2\right) \\ & + C\left(\sum_{j=1}^N \left(\mathbb{E}\left(\int_0^t \| h_j(s) \|_{L^2(D)}^2 ds\right)\right)^{1/2}\right)^2 + C\sum_{j=1}^N \mathbb{E}\left(\int_0^t \| h_j(s) \|_{L^2(D)}^2\right). \end{aligned} \tag{7}$$

## III. WEAK SOLUTION: EXISTENCE AND UNIQUENESS

In this section we will prove the existence and uniqueness of the weak solution to the system (1).

**Theorem 3.1** Let $0 < T < \infty$, $\phi_0 \in L^2(\Omega, \mathcal{F}_0; \chi)$, an $L^\infty(D)$-valued predictable process $u \in L^\infty(\Omega, \mathcal{F}_t; L^\infty(D \times (0,T)))$ and an $L^2(D)$-valued predictable process $f \in L^2(\Omega; L^2(D \times (0,T)))$ be given. Also, for each $j = 1, \cdots, N$, $f_j$, $g_j$ are given as in (2) and (3). Then there exist a weak solution to (1), Moreover, this solution is pathwise unique.

**Proof. Existence:**

Let us write

$$\left(y^0(t), y_t^0(t)\right) = \phi_0 = (\phi_0^1, \phi_0^2) \in L^2(\Omega, \mathcal{F}_0; \chi) \text{ for all } t \in [0,T] \tag{8}$$

and let $y^m$, $m = 1,2,\cdots$ be the solution of

$$\begin{aligned} & y_{tt} = \Delta y + uy^{m-1} + f + \sum_{j=1}^N (g_j + f_j y^{m-1})\frac{dW_j}{dt}, && \text{in } D \times (0,T), \\ & y = 0, && \text{on } \partial D \times [0,T], \\ & (y(0), y_t(0)) = \phi_0. \end{aligned} \tag{9}$$

As $\phi_0$ is $\chi$-valued $\mathcal{F}_0$-measurable, $u\phi_0^1$ is $L^2(D)$-valued and $\mathcal{F}_0$-measurable. Moreover, each $g_j + f_j\phi_0^1$ is an $L^2(D)$-valued predictable process, and

$$\| g_j + f_j\phi_0^1 \|_{L^2(\Omega; L^2(D\times(0,T)))} \le d_j + b_j\sqrt{T} \| \phi_0^1 \|_{L^2(\Omega; L^2(D))}, \tag{10}$$

where $d_j = \| g_j \|_{L^2(\Omega;L^2(D\times(0,T)))}$. If $(y^{m-1}, y_t^{m-1}) \in L^2(\Omega; C([0,T];\chi))$ is adapted to $\{\mathcal{F}_t\}$, $uy^{m-1}$ is an $L^2(D)$-valued predictable process. Also, each $g_j + f_j y^{m-1}$ is an $L^2(D)$-valued predictable process and

$$\| g_j + f_j y^{m-1} \|_{L^2(\Omega;L^2(D\times(0,T)))} \leq d_j + b_j \| y^{m-1} \|_{L^2(\Omega;L^2(D\times(0,T)))}. \tag{11}$$

Therefore, we can apply Lemma 2.1. By using Poincar$e'$'s inequality along with (3), (7) and boundedness of $u$, we have

$$\begin{aligned} &\mathbb{E}\left(\sup_{s\in[0,t]}\left(\| y_s^{m+1}(s) - y_s^m(s) \|_{L^2(D)}^2 + \| \nabla y^{m+1}(s) - \nabla y^m(s) \|_{L^2(D)}^2\right)\right) \\ &\leq C_T \mathbb{E}\left(\int_0^t \| \nabla y^m(s) - \nabla y^{m-1}(s) \|_{L^2(D)}^2 \, ds\right) \\ &+ C_T \mathbb{E}\left(\int_0^t \| y_s^m(s) - y_s^m(s) \|_{L^2(D)}^2 \, ds\right) \end{aligned} \tag{12}$$

For all $t \in [0,T]$, where $C_T$ is a positive constant independent of $m$. Let define

$$\Phi_m(t) = \mathbb{E}\left(\sup_{s\in[0,t]}\left(\| y_s^{m+1}(s) - y_s^m(s) \|_{L^2(D)}^2 + \| \nabla y^{m+1}(s) - \nabla y^m(s) \|_{L^2(D)}^2\right)\right)$$

For each $m = 1,2,\cdots$.

Then, from (12), we derive

$$\Phi_m(t) \leq C \int_0^t \Phi_{m-1}(s) ds \quad \text{for all } t \in [0,T] \text{ and all } m \geq 1, \tag{13}$$

where$C$ is a positive constant. By (7) and (10), we can find some positive constant $K$ such that

$$\Phi_0(t) \leq K \quad \text{forall} t \in [0,T]. \tag{14}$$

From (13) and (14), it follows that

$$\Phi_m(t) \leq \frac{KC^m t^m}{m!} \quad \text{for all } t \in [0,T]. \tag{15}$$

Thus,

$$\sum_{m=1}^{\infty} \sqrt{\Phi_m(T)} < \infty. \tag{16}$$

So, $\{(y^m, y_t^m)\}_{m=1}^{\infty}$ is a Cauchy sequence in $L^2(\Omega; C([0,T];\chi))$, and hence convergent. The limit of this sequence is a solution of (1).

**Uniqueness:**

Let $y_1$ and $y_2$ be two solutions of (1). For $i = 1,2$, define $k_i = uy_i$ and $h_{i,j} = f_j y_i$, for $j = 1,\cdots,N$. Let us treat $k_i$'s and $h_{i,j}$'s as given functions. Then by Lemma 2.1, there exists unique solution $\Psi_i$ to the following linear problem

$$\begin{aligned} &y_{tt} = \Delta y + k_i + f + \sum_{j=1}^N (g_j + h_{i,j}) \frac{dW_j}{dt}, && \text{in } D \times (0,T), \\ &y = 0, && \text{on } \partial D \times [0,T], \\ &(y(0), y_t(0)) = \phi_0. \end{aligned} \tag{17}$$

Moreover,

$$\Psi_i = y_i, \quad i = 1,2, \tag{18}$$

and, for all $t \in [0,T]$

$$\begin{aligned} &\mathbb{E}\left(\sup_{s\in[0,t]}\left(\| \partial_s \Psi_1(s) - \partial_s \Psi_2(s) \|_{L^2(D)}^2 + \| \nabla \Psi_1(s) - \nabla \Psi_2(s) \|_{L^2(D)}^2\right)\right) \\ &\leq C\left(\sum_{j=1}^N \left(\mathbb{E}\left(\int_0^t \| h_{1,j}(s) - h_{2,j}(s) \|_{L^2(G)}^2 \, ds\right)\right)^{\frac{1}{2}}\right)^2 \\ &+ C\mathbb{E}\left(\int_0^t \| k_1(s) - k_2(s) \|_{L^2(D)} \, ds\right)^2 \\ &+ C \sum_{j=1}^N \mathbb{E} \int_0^t \| h_{1,j}(s) - h_{2,j}(s) \|_{L^2(D)}^2 \, ds \end{aligned} \tag{19}$$

Thus, by (3), Poincar$e'$'s inequality, and (19), we have

$$\begin{aligned} &\mathbb{E}\left(\sup_{s\in[0,t]}\left(\| \partial_s y_1(s) - \partial_s y_2(s) \|_{L^2(D)}^2 + \| \nabla y_1(s) - \nabla y_2(s) \|_{L^2(D)}^2\right)\right) \\ &\leq C_T \mathbb{E}\left(\int_0^t \left(\| \partial_s y_1(s) - \partial_s y_2(s) \|_{L^2(D)}^2 + \| \nabla y_1(s) - \nabla y_2(s) \|_{L^2(D)}^2\right) ds\right) \end{aligned} \tag{20}$$

for all $t \in [0,T]$.

Applying Gronwall inequality, we have $y_1 = y_2$ for almost all $\omega \in \Omega$.

Following lemma gives the priori estimates to the solution of system (1).

**Lemma 3.1** Let $0 < T < \infty$, $\phi_0 \in L^2(\Omega, \mathcal{F}_0; \chi)$, an $L^\infty(D)$-valued predictable process $u \in L^\infty(\Omega, \mathcal{F}_t; L^\infty(D \times (0,T)))$ and an $L^2(D)$-valued predictable process $f \in L^2(\Omega; L^2(D \times (0,T)))$ be given. Also, for each $j = 1, \cdots, N$, $g_j$, $f_j$ are given as in (2) and (3). Then the weak solution $y$ to (1) satisfies the following inequality

$$\mathbb{E} \sup_{0 \leq t \leq T} \left( \| y \|^2_{H^1_0(D)} + \| y_t \|^2_{L^2(D)} \right) \leq C_T, \tag{21}$$

for some constant $C_T > 0$.

**Proof.** Proof is similar to the proof of Lemma 2.3 in [9]

## IV. OPTIMAL CONTROL AND OPTIMALITY SYSTEM

**Theorem 4.1**There exists an optimal control $\hat{u} \in \mathcal{U}$ which minimizes the cost functional (5).

**Proof.** Let $\{u^n\} \in L^\infty(\Omega, \mathcal{F}_t; L^\infty(D \times (0,T)))$ be the minimizing sequence such that

$$\lim_{n\to\infty} J(u^n) = \inf_{u \in \mathcal{U}} J(u).$$

Denote $y^n = y(u^n)$. By Lemma 3.1, we have

$$\mathbb{E} \sup_{0 \leq t \leq T} \left( \| y^n \|_{H^1_0(D)} + \| y^n_t \|_{L^2(D)} \right) \leq C_T.$$

As, this is a bounded sequence in a Banach space, there exists a convergent subsequence of $\{u^n\}$, again denoted by $\{u^n\}$ (for simplicity) converging to $u^* \in L^\infty(\Omega, \mathcal{F}_t; L^\infty(D \times (0,T)))$.

Hence, by weak compactness, there exists $y^* \in C(\Omega \times [0,T]; H^1_0(D))$ such that

$$\begin{aligned} y^n &\to y^* \quad \text{weakly in } L^\infty(\Omega \times [0,T]; H^1_0(D)), \\ y^n_t &\to y^*_t \quad \text{weakly in } L^\infty(\Omega \times [0,T]; L^2(D)), \\ u^n &\to u^* \quad \text{weakly in } L^2(\Omega \times D \times [0,T]). \end{aligned}$$

By using compactness results from [10], we have $y^n \to y^*$ in $L^\infty(\Omega \times [0,T]; L^2(D))$ and by weak formulation, we have

$$\langle y^n_t, v \rangle = \langle y^n_t(0), v \rangle - \int_0^t \langle \nabla y^n, \nabla v \rangle ds + \int_0^t \langle u^n y^n + f, v \rangle ds + \sum_{j=1}^N \int_0^t \langle g_j + f_j y^n, v \rangle dW_j.$$

for any $v \in H^1_0(\Omega)$ and a.e. $0 \leq t \leq T$ for almost. Also, as

$$\begin{aligned} y^n &\to y^* \quad \text{strongly in } L^2(\Omega \times D \times (0,T)), \\ u^n &\to u^* \quad \text{weakly in } L^2(\Omega \times D \times (0,T)), \end{aligned}$$

we have

$$y^n u^n \to y^* u^* \text{ weakly in } L^2(\Omega \times D \times (0,T)).$$

So, passing to the limit as $n \to \infty$ in the weak formulation, we have

$$\langle y^*_t, v \rangle = \langle y^*_t(0), v \rangle - \int_0^t \langle \nabla y^*, \nabla v \rangle ds + \int_0^t \langle u^* y^* + f, v \rangle ds + \sum_{j=1}^N \int_0^n \langle g_j + f_j y^*, v \rangle dW_j.$$

Thus $y^* = y(u^*)$ is the solution of system (1) with control $u^*$. Since

$$\begin{aligned} J(u^*) &\leq \lim_{n\to\infty} J(u^n) \\ &= \inf_{u \in \mathcal{U}} J(u). \end{aligned}$$

This implies that $h^*$ is an optimal control.

To obtain the optimality system, we require the adjoint problem of the stochastic wave problem (1). In general, we want to find a function $\Phi: \Omega \times D \times [0,T] \to \mathbb{R}$ with

$$\mathbb{E} \int_0^T \int_D |\Phi(\xi, t)|^2 d\xi dt < \infty$$

such that

$$\mathbb{E} \int_0^T \int_D \Phi \left[ \eta_{tt} - \Delta\eta - u\eta - \sum_{j=1}^N f_j \eta \frac{dW_j}{dt} \right] dx dt = \mathbb{E} \int_0^T \int_D (y - z) \eta dx dt. \tag{22}$$

For all $\eta: \Omega \times D \times [0,T] \to \mathbb{R}$ and all admissible controls $u \in \mathcal{U}$, where $y$ is the weak solution of the stochastic wave problem (1) and $z \in L^2(\Omega; C([0,T]; H^1_0(D)))$ is desired profile given in (5).

The solution $\Phi(\xi, t)$ of equation (22) is called the generalized solution of the adjoint problem of the stochastic wave problem (1), where the adjoint equation is

$$\Phi_{tt} = \Delta\Phi + u\Phi + (y - z) + \sum_{j=1}^N f_j \Phi \frac{dW_j}{dt}, \text{ in } D \times (0,T), \tag{23}$$

with the terminal and Dirichlet boundary conditions

$$\begin{aligned} &\Phi(x,T) = \Phi_t(x,T) = 0, \quad \forall\ x \in D, \\ &\Phi(x,t) = 0, \qquad \forall\ (x,t) \in \partial D \times [0,T]. \end{aligned} \tag{24}$$

The existence and uniqueness of the unique generalized solution to the adjoint problem follows from the Theorem 3.1. Furthermore, for the generalized solution $\Phi(x,t)$ of the adjoint problem, there exists a positive constant $C_T$ such that

$$\mathbb{E}\sup_{0\le t\le T}\left(\| \Phi \|^2_{H_0^1(D)} + \| \Phi_t \|^2_{L^2(D)}\right) \le C_T. \tag{25}$$

By $y_u$, we will mean a solution $y$ of (1) corresponding to the arbitrary control $u$. Let $u$ be an admissible control and $\epsilon > 0$ be sufficiently small such that $u+\epsilon l$ is also an admissible control. Let $\delta y$ denotes the difference between $y_{u+\epsilon l}$ and $y_u$, that is, $\delta y = y_{u+\epsilon l} - y_u$.

**Lemma 4.1**$\delta y$ is the weak solution of the initial and boundary value problem

$$\begin{aligned} &\frac{\partial^2}{\partial t^2}\delta y = \Delta\delta y + (u+\epsilon l)\delta y + \epsilon l y_u + \sum_{j=1}^{N} f_j \delta y \frac{dW_j}{dt}, && \text{in } D\times(0,T),\\ &\delta y(x,0) = \delta y_t(x,0) = 0, && \forall\ x\in D,\\ &\delta y(x,t) = 0, && \forall\ (x,t)\in\partial D\times[0,T]. \end{aligned} \tag{26}$$

**Proof.** By taking two solutions of this kind and taking their difference we get the system (26).

**Theorem 4.2**The G$\hat{a}$teaux derivative of the objective functional (5) is represented by

$$\delta J(u)(l) = \mathbb{E}\int_0^T \int_D l(\beta u + \Phi y_u) \tag{27}$$

for all admissible controls $l \in \mathcal{U}$.

**Proof.** We have

$$\begin{aligned} J(u+\epsilon l) - J(u) \quad &= \mathbb{E}\left(\frac{1}{2}\int_0^T \int_D (y_{u+\epsilon l} - z)^2 dxdt\right) + \mathbb{E}\left(\frac{\beta}{2}\int_0^T \int_D (u+\epsilon l)^2 dxdt\right)\\ &\quad -\mathbb{E}\left(\frac{1}{2}\int_0^T \int_D (y_u - z)^2 dxdt\right) - \mathbb{E}\left(\frac{\beta}{2}\int_0^T \int_D u^2 dxdt\right), \end{aligned}$$

Which simplifies to

$$\begin{aligned} J(u+\epsilon l) - J(u) \quad &= \mathbb{E}\left(\frac{1}{2}\int_0^T \int_D \delta y(y_{u+\epsilon l} + y_u - 2z) dxdt\right)\\ &\quad +\mathbb{E}\left(\frac{\beta}{2}\int_0^T \int_D (2u\epsilon l + \epsilon^2 l^2) dxdt\right). \end{aligned} \tag{28}$$

Replacing $\eta$ by $\delta y$ in (22) gives

$$\begin{aligned} &\mathbb{E}\int_0^T \int_D \Phi\left[\delta y_{tt} - \Delta\delta y - u\delta y - \sum_{j=1}^{N} f_j \delta y \frac{dW_j}{dt}\right] dxdt =\\ &\mathbb{E}\int_0^T \int_D (y-z)\delta y dxdt. \end{aligned} \tag{29}$$

Multiplying (26) with $\Phi$, integrating over $D\times[0,T]$ and taking expectation, we obtain

$$\begin{aligned} &\mathbb{E}\int_0^T \int_D \Phi\left[-\delta y_{tt} + \Delta\delta y + (u+\epsilon l)\delta y + \sum_{j=1}^{N} f_j \delta y \frac{dW_j}{dt}\right] dxdt =\\ &-\mathbb{E}\int_0^T \int_D \Phi\epsilon l y_u dxdt. \end{aligned} \tag{30}$$

Adding (29) and (30), we have

$$\mathbb{E}\int_0^T \int_D \Phi\epsilon l\delta y dxdt = \mathbb{E}\int_0^T \int_D (y-z)\delta y dxdt - \mathbb{E}\int_0^T \int_D \Phi\epsilon l y_u dxdt.$$

Thus

$$\mathbb{E}\int_0^T \int_D (y-z)\delta y dxdt = \mathbb{E}\left(\int_0^T \int_D [\Phi\epsilon l\delta y + \Phi\epsilon l y_u] dxdt\right).$$

Hence

$$\mathbb{E}\int_0^T \int_D \lim_{\epsilon\to 0}(y-z)\frac{\delta y}{\epsilon} dxdt = \lim_{\epsilon\to 0}\mathbb{E}\int_0^T \int_D (y-z)\frac{\delta y}{\epsilon} dxdt = \mathbb{E}\int_0^T \int_D \Phi l y_u dxdt.$$

Now,

$$\begin{aligned} \delta J(u)(l) \quad &= \lim_{\epsilon\to 0}\frac{J(u+\epsilon l) - J(u)}{\epsilon}\\ &= \lim_{\epsilon\to 0}\mathbb{E}\left(\frac{1}{2}\int_0^T \int_D \frac{\delta y(y_{u+\epsilon l} + y_u - 2z)}{\epsilon} dxdt + \frac{\beta}{2}\int_0^T \int_D (2ul + \epsilon l^2) dxdt\right)\\ &= \mathbb{E}\left(\int_0^T \int_D \lim_{\epsilon\to 0}\frac{\delta y}{\epsilon}(y-z) d\xi dt\right) + \mathbb{E}\int_0^T \int_D \beta ul dxdt. \end{aligned}$$

So,

$$\begin{aligned} \delta J(u)(l) \quad &= \mathbb{E}\int_0^T \int_D l y_u \Phi dxdt + \mathbb{E}\int_0^T \int_D \beta ul dxdt\\ &= \mathbb{E}\int_0^T \int_D l(y_u\Phi + \beta u) dxdt. \end{aligned} \tag{31}$$

Hence, the necessary optimality condition

$$\mathbb{E}(\beta u + \Phi y_u) \ge 0 \tag{32}$$

can be formulated.

By standard control argument, from inequality (32) we get

$$\beta u + \Phi y_u = 0 \ \mathbb{P} - \mathrm{a.s.} \tag{33}$$

The stochastic wave equation (1), adjoint problem (23), (24) and equality (33), forms the optimality system(OS).

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