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Secure Inverse Domination in the Join of Graphs

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Abstract: Let G be a connected simple graph and let D be a minimum dominating set of G. A dominating set $S \subseteq V(G) \setminus D$ is an inverse dominating set of G with respect to D. The set S is called a secure inverse dominating set of G if for every $u \in V(G) \setminus S$, there exists $v \in S$ such that $uv \in E(G)$ and the set $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of G. The secure inverse domination number of G, denoted by $\gamma_s^{(-1)}(G)$, is the minimum cardinality of a secure inverse dominating set of G. A secure inverse dominating set of cardinality$\gamma_s^{(-1)}(G)$ is called $\gamma_s^{(-1)} -$ set. In this paper, the researchers initiate a study of the concept of secure inverse domination in graphs and characterize the secure inverse dominating set in the join of two connected simple graphs.
Keywords: dominating sets, inverse dominating sets, join of two graphs, secure dominating sets, secure inverse dominating sets

I. INTRODUCTION

In [1], Claude Berge and Oystein Ore introduced the domination in graphs. Through the work of Cockayne and Hedetniemi in [2], domination in graphs became an area of study by many researchers [3, 4, 5, 6, 7, 8]. Secure domination in graphs was studied and introduced by E.J. Cockayne et.al [9, 10]. In [11] Enriquez and Canoy, introduced a variant of domination in graphs, the concept of secure convex domination in graphs. Some studies on secure domination in graphs were found in the papers [12, 13, 14, 15, 16, 17]. The inverse domination in graph was first found in the paper of Kulli [18] and can be read in [19, 20, 21, 22, 23, 24, 25]. In this paper, the researchers characterize the secure inverse dominating sets in the joinof two graphs and give some important results. For the general concepts, the reader may refer to [26].

Let $G = (V(G), E(G))$ be a connected simple graph and $v \in V(G)$. The neighborhood of v is the set $N_G(v) = N(v) = \{u \in V(G): uv \in E(G)\}$. If $S \subseteq V(G)$, then the *open neighborhood* of S is the set $N_G(S) = N(S) = \bigcup_{v \in S} N_G(v)$. The *closed neighborhood* of S is $N_G[S] = N[S] = S \cup N(S)$. A subset S of $V(G)$ is a *dominatingset* of G if for every $v \in (V(G)\setminus S)$,there exists $x \in S$such that $xv \in E(G)$, i.e., $N[S] = V(G)$. The *domination number*$\gamma(G)$of $G$is the smallest cardinality of a dominating set of G.

A dominating set S in G is called a *secure dominating set* in G if for every $u \in V(G)\setminus S$, there exists $v \in S \cap N_G(u)$ such that $(S\setminus\{v\}) \cup \{u\}$ is a dominating set. The minimum cardinality of secure dominating set is called the *secure domination number* of G and is denoted by $\gamma_s(G)$. A secure dominating set of cardinality$\gamma_s(G)$is called $\gamma_s - set$ of G.

Let D be a minimum dominating set in G. The dominating set $S \subseteq V(G)\setminus D$ is called an *inverse dominating set* with respect to D. The minimum cardinality of inverse dominating set is called an *inverse domination number* of $G$and is denoted by $\gamma^{-1}(G)$. An inverse dominating set of cardinality$\gamma^{-1}(G)$ is called $\gamma^{-1} - set$ of G. Motivated by the definition of secure and inverse domination in graphs, the researchers define a new domination parameter.

Let G be a connected simple graph and let D be a minimum dominating set in$G$. Then a dominating set $S \subseteq V(G)\setminus D$ is an inverse dominating set in G with respect to D. The set S is called a secure inverse dominating set in G if for every $u \in V(G)\setminus S$, there exists $v \in S$ such that $uv \in E(G)$ and the set $(S\setminus\{v\}) \cup \{u\}$ is a dominating set in G. The secure inverse domination number of G, denoted by $\gamma_s^{-1}(G)$, is the minimum cardinality of a secure inverse dominating set of G. A secure inverse dominating set of cardinality$\gamma_s^{-1}(G)$ is called $\gamma_s^{-1} - set$. In this paper, the study of secure inverse domination in graphs is initiated and some important results are given.

II. RESULTS

Remark 2.1 *Let D be a minimum dominating set of G. Then $V(G) \setminus D$ is a secure dominating set of G, that is, $V(G) \setminus D$ is a secure inverse dominating set of G.*

In Remark 2.1, $V(G) \setminus D$ can be an inverse secure dominating set of G if D is a secure dominating set of G. Hence, every inverse secure dominating set is a secure inverse dominating set, however, the converse is not always true. For example, in $P_5 = [x_1, x_2, \ldots, x_5]$, the set $D = \{x_1, x_4\}$ is a minimum dominating set of P_5 and $S = V(P_5) \setminus D = \{x_2, x_3, x_5\}$ is an inverse dominating set with respect to D. Since S is a secure dominating set, it follows that S is a secure inverse dominating set of P_5. However, it is not an inverse secure dominating set of G because D is not a secure dominating set of G. The following definitions are needed for the subsequent results.

Definition 2.2 *A nonempty subset S of $V(G)$, where G is any graph, is a clique in G if the graph $\langle S \rangle$ induced by S is complete.*

Definition 2.3 *The join of two graphs G and H is the graph $G + H$ with vertex-set $V(G + H) = V(G) \uplus V(H)$ and edge-set $E(G + H) = E(G) \uplus E(H) \cup \{uv: u \in V(G), v \in V(H)\}$.*

Remark 2.4 *If $\gamma(G) = 1$ or $\gamma(H) = 1$, then $\gamma(G + H) = 1$, otherwise, $\gamma(G + H) = 2$.*

Lemma 2.5 *Let G be connected non-complete graphs. If D is a minimum dominating set of G with $|D| \leq 2$ and S is an inverse dominating set of G with respect to D, then a subset $S \subseteq V(G + H) \setminus D$ is a secure inverse dominating set of $G + H$.*

Proof: Suppose that D is a minimum dominating set of G with $|D| \leq 2$ and S is an inverse dominating set of G with respect to D. Then $S \subseteq V(G + H) \setminus D$ is an inverse dominating set of $G + H$with respect to D. Clearly, if $S = V(G + H) \setminus D$, then S is a secure inverse dominating set of $G + H$with respect to D. Now, let $S \subset V(G + H) \setminus D$and consider the following cases.

Case 1. If$S = V(G) \setminus D$, then let $u \in V(G) \setminus S = D \subset V(G + H) \setminus S$. There exists $v \in S$such that $uv \in E(G + H)$ and $(S \setminus \{v\}) \cup \{u\}$.

Subcase 1. If $D = \{u\}$, then $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of $G + H$. Hence, S is a secure dominating set of $G + H$.

Subcase 2. If $D = \{u', u''\}$, then $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of $G + H$for all $u \in D$. Hence, S is a secure dominating set of $G + H$.

In either case, S is a secure dominating set of $G + H$. Accordingly, S is a secure inverse dominating set of $G + H$.

Case 2. If$S \neq V(G) \setminus D$, $S \subset V(G) \setminus D$. Note that S is an inverse dominating set of $G + H$. Thus, for all $u \in V(G) \setminus S \subset V(G + H) \setminus S$, there exists $v \in S$such that$uv \in E(G + H)$ and $(S \setminus \{v\}) \cup \{u\}$.

Subcase 1. If $D = \{u\}$, then $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of $G + H$. Hence, S is a secure dominating set of $G + H$.

Subcase 2. If $D = \{u', u''\}$, then $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of $G + H$ for all $u \in D$. Hence, S is a secure dominating set of $G + H$.

In either case, S is a secure dominating set of $G + H$. Accordingly, S is a secure inverse dominating set of $G + H$. □

Lemma 2.6 *Let G and H be connected non-complete graphs. If D is a minimum dominating set of G with $|D| \leq 2$ and $S = V(H)$ or S is a secure dominating set of H, then a subset $S \subseteq V(G + H) \setminus D$ is a secure inverse dominating set of $G + H$.*

Proof: Suppose that D is a minimum dominating set of G with $|D| \leq 2$ and $S = V(H)$ or S is a secure dominating set of H.

Case 1. If$S = V(H)$, then S is an inverse dominating set of $G + H$ with respect to a minimum dominating set D of G and of $G + H$ is clear. Since for every $u \in V(G + H) \setminus S = V(G)$, there exists $v \in S$such that $uv \in E(G + H)$ and $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of $G + H$, S is a secure dominating set of $G + H$. Thus, S is a secure inverse dominating set of $G + H$.

Case 2. If S is a secure dominating set of H, then$S \subset V(H)$. Further, $S \subseteq V(G + H) \setminus D$ is an inverse dominating set of $G + H$ with respect to a minimum dominating set D of G and of $G + H$. Now, for every $u \in V(G + H) \setminus S = V(G) \cup (V(H) \setminus S)$, that is, $u \in V(G) \cup (V(H) \setminus S)$. If $u \in V(G)$, then there exists $v \in S$such that $uv \in E(G + H)$ and $(S \setminus \{v\}) \cup \{u\}$ is a dominatingset of $G + H$ (since H is a connected non-complete graph, $|S| \geq 2$). Hence, S is a secure dominating set of $G + H$, that is, S is a secure inverse dominating set of $G + H$. If $u \in V(H) \setminus S$, then there exists $v \in S$ such that $uv \in E(H) \subset E(G + H)$ (since S is a dominating set of H) and $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of H (since S is a secure dominating set of H) and of $G + H$. Hence, S is a secure inverse dominating set of $G + H$. □

Lemma 2.7 *Let G and H be connected non-complete graphs. If D ($|D| \leq 2$) is a minimum dominating set of H and S is an inverse dominating set of H with respect to D, then a subset $S \subseteq V(G + H) \setminus D$, is a secure inverse dominating set of $G + H$.*

Proof: Suppose that D is a minimum dominating set of G with $|D| \leq 2$ and S is an inverse dominating set of H with respect to D. Then $S \subseteq V(G+H) \setminus D$ is an inverse dominating set of $G+H$ with respect to D. Clearly, if$S = V(G+H) \setminus D$, then S is a secure inverse dominating set of $G+H$ with respect to D. Now, let $S \subset V(G+H) \setminus D$ and consider the following cases.

Case 1. If $S = V(H) \setminus D$, then let $u \in V(H) \setminus S = D \subset V(G+H) \setminus S$. There exists $v \in S$such that$uv \in E(G+H)$ and $(S \setminus \{v\}) \cup \{u\}$.

Subcase 1. If $D = \{u\}$, then $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of $G+H$. Hence, S is a secure dominating set of $G+H$.

Subcase 2. If $D = \{u', u''\}$ then $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of $G+H$ for all $u \in D$. Hence, S is a secure dominating set of $G+H$.

In either case, S is a secure dominating set of $G+H$. Accordingly, S is a secure inverse dominating set of $G+H$.

Case 2. If $S \neq V(H) \setminus D$, then$S \subset V(H) \setminus D$. Note that S is an inverse dominating set of $G+H$. Thus, for all $u \in V(H) \setminus S \subset V(G+H) \setminus S$, there exists $v \in S$ such that $uv \in E(G+H)$ and $(S \setminus \{v\}) \cup \{u\}$.

Subcase 1. If $D = \{u\}$, then$(S \setminus \{v\}) \cup \{u\}$ is a dominating set of $G+H$. Hence, S is a secure dominating set of $G+H$.

Subcase 2. If $D = \{u', u''\}$then $(S \setminus \{v\}) \cup \{u\}$is a dominating set of $G+H$ for all $u \in D$. Hence, S is a secure dominating set of $G+H$.

In either case, S is a secure dominating set of $G+H$. Accordingly, S is a secure inverse dominating set of $G+H$.□

Lemma 2.8 *Let G and H be connected non-complete graphs. If D ($|D| \leq 2$) is a minimum dominating set of H and $S = V(G)$ or $S$is a secure dominating set of G, then a subset $S \subseteq V(G+H) \setminus D$ is a secure inverse dominating set of $G+H$.*

Proof: Suppose that D is a minimum dominating set of G with $|D| \leq 2$ and $S = V(G)$or S is a secure dominating set of G. Then $S \subseteq V(G)$.

Case 1. If $S = V(G)$, then S is an inverse dominating set of $G+H$ with respect to a minimum dominating set D of H and of $G+H$. Since for every $u \in V(G+H) \setminus S = V(H)$, there exists $v \in S$ such that $uv \in E(G+H)$ and $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of $G+H$. Hence, S is a secure dominating set of $G+H$, that is, S is a secure inverse dominating set of $G+H$.

Case 2. If $S \neq V(G)$, then $S \subset V(G)$. Further, S is an inverse dominating set of $G+H$ with respect to a minimum dominating set D of H and of $G+H$. Now, for every $u \in V(G+H) \setminus S = V(H) \cup (V(G) \setminus S)$, $u \in V(H) \cup (V(G) \setminus S)$. If $u \in V(H)$, then there exists $v \in S$ such that$uv \in E(G+H)$ and $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of $G+H$ (since $G$is a connected non-complete graph, $|S| \geq 2$). Hence, S is a secure dominating set of $G+H$, that is, S is a secure inverse dominating set of $G+H$. If $u \in V(G) \setminus S$, then there exists $v \in S$ such that $uv \in E(G) \subset E(G+H)$ (since S is a dominating set of G) and $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of G (since S is a secure dominating set of G) and of$G+H$. Hence, S is a secure inverse dominating set of $G+H$.□

Lemma 2.9 *Let $G$and H be connected non-complete graphs. If $D = D_G \cup D_H$ where $D_G = \{v\} \subset V(G), D_H = \{w\} \subset V(H), \gamma(G) \neq 1, \gamma(H) \neq 1$, and $S = V(G) \setminus D_G$ or $S = V(H) \setminus D_H$, then a subset $S \subseteq V(G+H) \setminus D$, is a secure inverse dominating set of $G+H$.*

Proof: Suppose that $D = D_G \cup D_H$where$D_G = \{v\} \subset V(G), D_H = \{w\} \subset V(H), \gamma(G) \neq 1$, and $\gamma(H) \neq 1$. Then $D = \{v, w\}$ is a minimum dominating set of $G+H$ and $S \subseteq V(G+H) \setminus D$ is an inverse dominating set of $G+H$ with respect to D.

Case 1. If $S = V(G) \setminus D_G$, then S is clearly a secure dominating set of G. Now, $V(G+H) \setminus S \neq \emptyset$, let $u \in V(G+H) \setminus S$. If $u \in V(G) \setminus S$, then there exists$x \in S$ such that $ux \in E(G) \subset E(G+H)$and $(S \setminus \{x\}) \cup \{u\}$ is a dominating set of G (since S is a secure dominating set of G) and of $G+H$. Hence, S is a secure dominating set of $G+H$, that is,S is a secure inverse dominating set of $G+H$. If $u \in V(H)$, then there exists $x \in S \subset V(G)$ such that $xu \in E(G+H)$ and $(S \setminus \{x\}) \cup \{u\}$is a dominating set of$G+H$. Hence, S is a secure dominating set of $G+H$, that is,S is a secure inverse dominating set of $G+H$.

Case 2. If$S = V(H) \setminus D_H$, then S is clearly a secure dominating set of H and an inverse dominating set of $G+H$ with respect to D. Now, $V(G+H) \setminus S \neq \emptyset$, let $u \in V(G+H) \setminus S$. If $u \in V(H) \setminus S$, then there exists $x \in S$ such that $ux \in E(H) \subset E(G+H)$and $(S \setminus \{x\}) \cup \{u\}$ is a dominating set of H (since S is a secure dominating set of H) and of $G+H$. Hence, S is a secure dominating set of $G+H$, that is,S is a secure inverse dominating set of $G+H$. If $u \in V(G)$, then there exists $x \in S \subset V(H)$ such that $xu \in E(G+H)$ and $(S \setminus \{x\}) \cup \{u\}$is a dominating set of $G+H$. Hence, S is a secure dominating set of $G+H$, that is,S is a secure inverse dominating set of $G+H$.□

Lemma 2.10 *Let G and H be connected non-complete graphs. If $D = D_G \cup D_H$ where $D_G = \{v\} \subset V(G), D_H = \{w\} \subset V(H), \gamma(G) \neq 1, \gamma(H) \neq 1,$ and $S \subset V(G) \setminus D_G$ is a secure dominating set of G or $S \subset V(H) \setminus D_H$ is a secure dominating set of H, then a subset $S \subseteq V(G+H) \setminus D$, is a secure inverse dominating set of $G+H$.*

Proof: Suppose that $D = D_G \cup D_H$ where $D_G = \{v\} \subset V(G), D_H = \{w\} \subset V(H), \gamma(G) \neq 1, and\ \gamma(H) \neq 1$. Then $D = \{v, w\}$ is a minimum dominating set of $G+H$ and $S \subseteq V(G+H) \setminus D$ is an inverse dominating set of $G+H$ with respect to D.

Case 1. If $S \subset V(G) \setminus D_G$ is a secure dominating set of G, then for every $u \in (V(G) \setminus D_G) \setminus S$, there exists $x \in S$ such that $ux \in E(G)$ and $(S \setminus \{x\}) \cup \{u\}$ is a dominating set of G and of $G+H$. Thus, S is a secure inverse dominating set of $G+H$.

Case 2. If $S \subset V(H) \setminus D_H$ is a secure dominating set of H, then for every $u \in (V(H) \setminus D_H) \setminus S$, there exists $x \in S$ such that $ux \in E(H)$ and $(S \setminus \{x\}) \cup \{u\}$ is a dominating set of H and of $G+H$. Thus, S is a secure inverse dominating set of $G+H$.□

Lemma 2.11 *Let G and H be connected non-complete graphs. If $D = D_G \cup D_H$ where $D_G = \{v\} \subset V(G), D_H = \{w\} \subset V(H)$ and either D_G or D_H is a dominating set, and $S = S_G \cup S_H$ where $S_G \subset V(G)$ and $S_H \subset V(H)$ and $S_G = \{z\}$ is a dominating set of G and $S_H = \{x\}$ is a dominating set of H, then a subset $S \subseteq V(G+H) \setminus D$ is a secure inverse dominating set of $G+H$.*

Proof: Suppose that $D = D_G \cup D_H$ where $D_G = \{v\} \subset V(G)$ or $D_H = \{w\} \subset V(H)$ is a dominating set of G or H. Then D is a minimum dominating set of $G+H$. If $S = S_G \cup S_H$, where $S_G = \{z\} \subset V(G)$ and $S_H = \{x\} \subset V(H)$ are dominating sets in G and H respectively, then $S = \{z, x\} \subset V(G+H) \setminus D$ is an inverse dominating set of $G+H$ with respect to a minimum dominating set D. Let $u \in V(G+H) \setminus S$. If $u \in V(G) \setminus S_G$, then $uz \in E(G) \subset E(G+H)$ and $(S \setminus \{z\}) \cup \{u\} = \{x, u\}$ is a dominating set of $G+H$. Hence, S is a secure dominating set of $G+H$. If $u \in V(H) \setminus S_H$, then $ux \in E(H) \subset E(G+H)$ and $(S \setminus \{x\}) \cup \{u\} = \{z, u\}$ is a dominating set of $G+H$. Hence, S is a secure dominating set of $G+H$.□

Lemma 2.12 *Let G and H be connected non-complete graphs. If $D = D_G \cup D_H$ where $D_G = \{v\} \subset V(G), D_H = \{w\} \subset V(H)$, $S = S_G \cup S_H$ where $S_G \subseteq V(G) \setminus D_G$ and $S_H \subseteq V(H) \setminus D_H$, $\gamma(G) \neq 1, \gamma(H) \neq 1$, $S_G = \{z\}$ and $\langle (V(H) \setminus N_H[S_H]) \rangle$ is a clique in H, where $|S_H| \geq 2$, then a subset $S \subseteq V(G+H) \setminus D$ is a secure inverse dominating set of $G+H$.*

Proof: Suppose that $D = D_G \cup D_H$ where $D_G = \{v\} \subset V(G), D_H = \{w\} \subset V(H)$. If $\gamma(G) \neq 1$ and $\gamma(H) \neq 1$, then $D = \{v, w\}$ is a minimum dominating set of $G+H$. Since $S = S_G \cup S_H$ where $S_G \subseteq V(G) \setminus D_G$ and $S_H \subseteq V(H) \setminus D_H$, it follows that $S \subseteq V(G+H) \setminus D$ is an inverse dominating set of $G+H$ with respect to D. Let $u \in V(G+H) \setminus S$.

If $u \in V(G) \setminus S_G$ and $u \in N_G[S_G] = N_G[\{z\}]$, then $uz \in E(G) \subset E(G+H)$ and $(S \setminus \{z\}) \cup \{u\} = S_H \cup \{u\}$ is a dominating set of $G+H$, that is, S is a secure dominating set of $G+H$.

If $u \in V(G) \setminus S_G$ and $u \notin N_G[\{z\}]$, then there exists $y \in S_H$ such that $uy \in E(G+H)$ and $(S \setminus \{y\}) \cup \{u\} = (\{z\} \cup S_H \setminus \{y\}) \cup \{u\}$ is a dominating set of $G+H$ (since $|S_H| \geq 2$), that is, S is a secure dominating set of $G+H$.

If $u \in V(H) \setminus S_H$ and $u \in N_H[S_H]$, then there exists $y \in S_H$ such that $uy \in E(H) \subset E(G+H)$ and $(S \setminus \{y\}) \cup \{u\} = (S_G \cup S_H \setminus \{y\}) \cup \{u\}$ is a dominating set of $G+H$, that is, S is a secure dominating set of $G+H$.

If $u \in V(H) \setminus S_H$ and $u \notin N_H[S_H]$, then $u \in V(H) \setminus N_H[S_H]$. Further, $zu \in E(G+H)$ and $(S \setminus \{z\}) \cup \{u\} = S_H \cup \{u\}$. Since $u \in V(H) \setminus N_H[S_H]$ and $\langle V(H) \setminus N_H[S_H] \rangle$ is a clique in H, it follows that $S_H \cup \{u\}$ is a dominating set of H and $G+H$, that is, S is a secure dominating set of $G+H$.

Accordingly, S is a secure inverse dominating set of $G+H$.□

Lemma 2.13 *Let G and H be connected non-complete graphs. If $D = D_G \cup D_H$ where $D_G = \{v\} \subset V(G), D_H = \{w\} \subset V(H)$, $S = S_G \cup S_H$ where $S_G \subset V(G) \setminus D_G$ and $S_H \subset V(H) \setminus D_H$, $\gamma(G) \neq 1, \gamma(H) \neq 1$, $\langle (V(G) \setminus$ NGSG) is a clique in G, where SG≥2 and SH=x, then a subset S⊆VG+H\D, is a secure inverse dominating set of $G+H$.*

Proof: Suppose that $D = D_G \cup D_H$ where $D_G = \{v\} \subset V(G)$, $D_H = \{w\} \subset V(H)$, $\gamma(G) \neq 1$, and $\gamma(H) \neq 1$,. Then $D = \{v, w\}$ is a minimum dominating set of $G+H$. Since $S = S_G \cup S_H$ where $S_G \subseteq V(G) \setminus D_G$ and $S_H \subseteq V(H) \setminus D_H$, it follows that $S \subseteq V(G+H) \setminus D$ is an inverse dominating set of $G+H$ with respect to D. Let $u \in V(G+H) \setminus S$.

If $u \in V(H) \setminus S_H$ and $u \in N_H[S_H] = N_H[\{x\}]$, then $ux \in E(H) \subset E(G+H)$ and $(S \setminus \{x\}) \subset \{u\} = S_G \cup \{u\}$ is a dominating set of $G+H$, that is, S is a secure dominating set of $G+H$.

If $u \in V(H) \setminus S_H$ and $u \notin N_H[\{x\}]$, then there exists $y \in S_G$ such that $uy \in E(G+H)$ and $(S \setminus \{y\}) \cup \{u\} = \left((S_G \setminus \{y\}) \cup \{x\}\right) \cup \{u\}$ is a dominating set of $G+H$ (since $|S_G| \geq 2$), that is, S is a secure dominating set of $G+H$.

If $u \in V(G)$ and $u \in N_G[S_G]$, then there exists $y \in S_G$ such that $uy \in E(G) \subset (G+H)$ and $(S \setminus \{y\}) \cup \{u\} = \big((S_G \setminus \{y\}) \cup \{x\}\big) \cup \{u\}$ is a dominating set of $G+H$, that is, S is a secure dominating set of $G+H$.

If $u \in v(G) \setminus S_G$ and $u \notin N_G[S_G]$, then$u \in V(G) \setminus N_G[S_G]$. Further, $ux \in E(G+H)$ and $(S \setminus \{x\}) \cup \{u\} = S_G \cup \{u\}$. Since $u \in V(G) \setminus N_G[S_G]$ and $\langle V(G) \setminus N_G[S_G] \rangle$ is a clique in G, it follows that $S_G \cup \{u\}$ is a dominating set of G and $G+H$, that is, S is a secure dominating set of $G+H$.

Accordingly, S is a secure inverse dominating set of $G+H$.□

Lemma 2.14 *Let $G$and H be connected non-complete graphs. If $D = D_G \cup D_H$ where $D_G = \{v\} \subset V(G)$, $D_H = \{w\} \subset V(H)$,$\gamma(G) \neq 1$, $\gamma(H) \neq 1$, $S_G = \{z\}$ and $\langle (V(G) \setminus N_G[S_G]) \rangle$is a clique in G, and$S_H = \{x\}$ and $\langle (V(H) \setminus N_H[S_H]) \rangle$ is a clique in H, then a subset $S \subseteq V(G+H) \setminus D$, is a secure inverse dominating set of $G+H$.*

Proof: Suppose that $D = D_G \cup D_H$ where $D_G = \{v\} \subset V(G)$, $D_H = \{w\} \subset V(H)$, $\gamma(G) \neq 1$, and $\gamma(H) \neq 1$. Then $D = \{v, w\}$ is a minimum dominating set of $G+H$. If $S_G = \{z\}$ and $\langle (V(G) \setminus N_G[S_G]) \rangle$ is a clique in G, $S_H = \{x\}$ and $\langle (V(H) \setminus N_H[S_H]) \rangle$ is a clique in H, then $S = \{z, x\} \subset V(G+H) \setminus D$ is a dominating set of $G+H$, that is,$S \subset V(G+H) \setminus D$ is an inverse dominating set of $G+H$ with respect to D. Let $u \in V(G+H) \setminus S$.

Case 1. If $u \in V(G) \setminus S_G$, then $ux \in E(G+H)$. If $u \in V(G) \setminus N_G[S_G]$, then $(S \setminus \{x\}) \cup \{u\} = \{z, u\} \subset V(G)$ is a dominating set of G since $\langle V(G) \setminus N_G[S_G] \rangle$ is a clique in G, that is, $u$dominates $\langle V(G) \setminus N_G[S_G] \rangle$ and z dominates $N_G[S_G]$. Thus,$(S \setminus \{x\}) \cup \{u\} = \{z, u\}$ is a dominating set of $G+H$. If $u \notin V(G) \setminus N_G[S_G]$, then $u \in N_G[S_G]$, that is, $(S \setminus \{z\}) \cup \{u\} = \{x, u\}$ is a dominating set of G and of $G+H$. Thus, S is a secure dominating set of $G+H$, that is, S is a secure inverse dominating set of $G+H$.

Case 2. If $u \in V(H) \setminus S_H$, then $uz \in E(G+H)$. If $u \in V(H) \setminus N_H[S_H]$, then $(S \setminus \{z\}) \cup \{u\} = \{x, u\}$ is a dominating set of H since $\langle V(H) \setminus N_H[S_H] \rangle$ and x dominates $N_H[S_H]$. Thus $(S \setminus \{z\}) \cup \{u\} = \{x, u\}$ is a dominating set of $G+H$. If $u \notin V(H) \setminus N_H[S_H]$, then $u \in N_H[S_H]$, that is, $(S \setminus \{x\}) \cup \{u\} = \{z, u\}$ is a dominating set of H and of $G+H$. Thus, S is a secure dominating set of $G+H$, that is, S is a secure inverse dominating set of $G+H$. □

Lemma 2.15 *Let $G$and H be connected non-complete graphs. If $D = D_G \cup D_H$where $D_G = \{v\} \subset V(G), D_H = \{w\} \subset V(H)$, $S = S_G \cup S_H$ where $S_G \subseteq V(G) \setminus D_GandS_H \subseteq V(H) \setminus D_H$, $\gamma(G) \neq 1, \gamma(H) \neq 1$, $|S_G| \geq 2$, and $|S_H| \geq 2$, then a subset $S \subseteq V(G+H) \setminus D$ is a secure inverse dominating set of $G+H$.*

Proof: Suppose that $D = D_G \cup D_H$ where $D_G = \{v\} \subset V(G)$, $D_H = \{w\} \subset V(H)$, $\gamma(G) \neq 1$, and $\gamma(H) \neq 1$. Then $D = \{v, w\}$ is a minimum dominating set of$G+H$. If $S = S_G \cup S_H$where $S_G \subseteq V(G) \setminus D_GandS_H \subseteq V(H) \setminus D_H$, $|S_G| \geq 2$, and$|S_H| \geq 2$, then S is an inverse dominating set of $G+H$ with respect to a minimum dominating set D of $G+H$. Let $u \in V(G+H) \setminus S$. If $u \in V(G) \setminus S_G$, then there exists $x \in S_H \subset S$ such that $ux \in E(G+H)$ and $(S \setminus \{x\}) \cup \{u\}$ is a dominating set of $G+H$(since $|S_H| \geq 2$). Similarly, if $u \in V(H) \setminus S_H$, then there exists $x \in S_G \subset S$ such that $ux \in E(G+H)$ and $(S \setminus \{x\}) \cup \{u\}$ is a dominating set of $G+H$(since $|S_G| \geq 2$). Thus, S is a secure dominating set of $G+H$, that is, S is a secure inverse dominating set of $G+H$. □

The following result is the characterization of the secure inverse dominating set in the join of two graphs.

Theorem 2.16 *Let G and H be connected non-complete graphs. Then a subset$S \subseteq V(G+H) \setminus D$, is a secure inverse dominating set of $G+H$ if and only if one of the following statements holds.*

(i) D is a minimum dominating set of G with $|D| \leq 2$ and
- *a) S is an inverse dominating set of G with respect to D, or*
- *b) $S = V(H)$or S is a secure dominating set of H.*

(ii)D ($|D| \leq 2$) is a minimum dominating set of H and
- *a) S is an inverse dominating set of H with respect to D, or*
- *b) $S = V(G)$ or S is a secure dominating set of G.*

(iii) $D = D_G \cup D_H$where $D_G = \{v\} \subset V(G)$, $D_H = \{w\} \subset V(H)$, $\gamma(G) \neq 1, \gamma(H) \neq 1$, and
- *a) $S = V(G) \setminus D_G$ or $S = V(H) \setminus D_H$, or*
- *b) $S \subset V(G) \setminus D_G$ is a secure dominating set of$G$ or $S \subset V(H) \setminus D_H$is a secure dominating set of H.*

(iv) $D = D_G \cup D_H$where $D_G = \{v\} \subset V(G)$, $D_H = \{w\} \subset V(H)$, either $D_G$or $D_H$is a dominating set, $S = S_G \cup S_H$where$S_G \subset V(G)$ and $S_H \subset V(H)$, and $S_G = \{z\}$ is a dominating set of G and $S_H = \{x\}$is a dominating set of H.

(v) $D = D_G \cup D_H$where $D_G = \{v\} \subset V(G)$ and $D_H = \{w\} \subset V(H)$, and $S = S_G \cup S_H$ where $S_G \subset V(G)$ and$S_H \subset V(H)$,$\gamma(G) \neq 1$, $\gamma(H) \neq 1$, and
- *a) $S_G = \{z\}$and $\langle (V(H) \setminus N_H[S_H]) \rangle$ is a clique in H, or*
- *b) $\langle (V(G) \setminus N_G[S_G]) \rangle$ is a clique in G, $S_H = \{x\}$, or*

c) $S_G = \{z\}$*and* $\langle(V(G) \setminus N_G[S_G])\rangle$ *is a clique in* G, $S_H = \{x\}$*and* $\langle(V(H) \setminus N_H[S_H])\rangle$ *is a clique in* H, *or*

d) $|S_G| \geq 2$ *and* $|S_G| \geq 2$.

Proof: Suppose that S is a secure inverse dominating set of $G + H$. Consider the following cases.

Case 1. Suppose that $D \cap V(H) = \emptyset$. Then $D \subseteq V(G)$. If $D = V(G)$, then $|D| \geq 3$ since G is a non-complete graph. This will contradict with the definition of D as a minimum dominating set of $G + H$(see Remark 2.4). Thus, D must be a minimum dominating set of G with $|D| \leq 2$. Since G is connected non-complete graph, $V(G) \setminus D \neq \emptyset$. Let $S \subseteq V(G) \setminus D$, that is, S is an inverse dominating set of G with respect to D. This shows statement (i)a). If $S \subseteq V(H)$, then $S = V(H)$or $S \subset V(H)$. Since S is a secure inverse dominating set of $G + H$, it follows that S is a secure dominating set of H. This shows statement (i)b).

Case 2. Suppose that $D \cap V(G) = \emptyset$. Then $D \subseteq V(H)$. If $D = V(H)$, then $|D| \geq 3$ since H is a non-complete graph. This will contradict with the definition of D as minimum dominating set of $G + H$ (see Remark 2.4). Thus, Dmust be a minimum dominating set of $H$with $|D| \leq 2$. Since H is connected non-complete graph, $V(H) \setminus D = \emptyset$. If $S \subset V(H)$, then $S = V(H) \setminus D$, that is, S is an inverse dominating set of $H$with respect to D. This shows statement (ii)a). If $S \subseteq V(G)$, then $S = V(G)$ or $S \subset V(G)$. Since, S is a secure inverse dominating set of $G + H$, it follows that S is a secure dominating set of G. This shows statement (ii)b).

Case 3. Suppose that $D \cap V(G) \neq \emptyset$and $D \cap V(H) \neq \emptyset$. Let $D_G = D \cap V(G)$ and $D_H = D \cap V(H)$. Then

$$\begin{aligned} D_G \cup D_H &= \big(D \cap V(G)\big) \cup \big(D \cap V(H)\big) \\ &= D \cap \big(V(G) \cup V(H)\big) \\ &= D \cap V(G + H) \\ &= D. \end{aligned}$$

By Remark 2.4, the minimum dominating set of $G + H$ is either 1 or 2. Let $D_G = \{v\}$ and $D_H = \{w\}$.

Subcase 1. If $S \cap V(H) = \emptyset$, then $S \subseteq V(G) \setminus D_G$. If $S = V(G) \setminus D_G$, then this satisfies statement (iii)a). If $S \subset V(G) \setminus D_G$, then statement (iii)b) is satisfied since S is a dominating set of $G + H$, that is, S is a dominating set of G.

Subcase 2. If $S \cap V(G) = \emptyset$, then $S \subseteq V(H) \setminus D_H$. If $S = V(H) \setminus D_H$, then this satisfies statement (iii)a). If $S \subset V(H) \setminus D_H$, then statement (iii)b) is satisfied since S is a dominating set of $G + H$, or S is a dominating set of H.

Subcase 3. If $S \cap V(G) \neq \emptyset$ and$S \cap V(H) \neq \emptyset$, then let $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$, that is, $S_G \subset V(G)$ and $S_H \subset V(H)$. Now,

$$\begin{aligned} S_G \cup S_H &= \big(S \cap V(G)\big) \cup (S \cap (V(H)) \\ &= S \cap \big(V(G) \cup V(H)\big) \\ &= S \cap V(G + H) \\ &= S. \end{aligned}$$

Thus, $S = S_G \cup S_H$ where $S_G \subset V(G)$ and $S_H \subset V(H)$.

If $S_G = \{z\}$ is a dominating set of G and $S_H = \{x\}$ is a dominating set of H, then the proofof statement (iv) is done.

If $S_G = \{z\}$ and S_H is not a dominating set of H with $|V(H)| \geq 2$. Then there exists $u \in V(H) \setminus S_H$ such that, $xu \notin E(H)$ for all $x \in S_H$. Since $S = S_G \cup S_H$is a secure dominating set of $G + H$, for all $u \in V(G + H) \setminus S$, there exists $v \in S$such that $uv \in E(G + H)$ and $(V(G + H) \setminus \{v\}) \cup \{u\}$ is a dominating set of $G + H$. This is clearly true if $v = z$, or $v \in S_H$ and $u \in N_H[v]$. However, since S_H is not a dominating set of H, if $v \in S_H$ and$u \notin N_H[v]$, then $\{u\} \subseteq V(H) \setminus N_H[v]$ must be a dominating set of $\langle V(H) \setminus N_H[v]\rangle$ for all $u \in V(H) \setminus N_H[v]$. This implies that the induced subgraph $\langle V(H) \setminus N_H[v]\rangle$ of $V(H) \setminus N_H[v]$ is a clique in H. This satisfies(v)a).

If $S_H = \{x\}$ and S_G is not a dominating set of G with $|V(H)| \geq 2$. Then there exists $u \in V(G) \setminus S_G$ such that $zu \notin E(H)$ for all $z \in S_G$. Since $S = S_G \cup S_H$is a secure dominating set of $G + H$, for all $u \in V(G + H) \setminus S$, there exists $v \in S$such that $uv \in E(G + H)$ and $(V(G + H) \setminus \{v\}) \cup \{u\}$ is a dominating set of $G + H$. This is clearly true if $v = x$, or $v \in S_G$ and $u \in N_G[v]$. However, since S_G is not a dominating set of G, if $v \in S_G$ and$u \notin N_G[v]$, then $\{u\} \subseteq V(G) \setminus N_G[v]$ must be a dominating set of $\langle V(G) \setminus N_G[v]\rangle$ for all $u \in V(G) \setminus N_G[v]$. This implies that the induced subgraph $\langle V(G) \setminus N_G[v]\rangle$ of $V(G) \setminus N_G[v]$ is a clique in G. This satisfies (v)b).

If $S_G = \{z\}$ and $S_H = \{x\}$, then S_G and S_H are not dominating sets of G and Hrespectively (since $\gamma(G) \neq 1$ and $\gamma(H) \neq 1$). Thus, there exists $u \in V(G) \setminus S_G$ such that $zu \notin E(G)$ for all $z \in S_G$and there exists $u \in V(H) \setminus S_H$ such that $xu \notin E(H)$ for all $x \in S_H$. Since $S = S_G \cup S_H$ is a secure dominating set of$G + H$, for all $u \in V(G + H) \setminus S$, there exists $v \in S$such that $uv \in E(G + H)$ and $(V(G + H) \setminus \{v\}) \cup \{u\}$ is a dominating set of $G + H$. Since $S_G$is not a dominating set of G, if $v = z$ and $u \notin N_G[v]$, then $\{u\} \subseteq V(G) \setminus N_G[v]$ must be a dominating set of $\langle V(G) \setminus N_G[v] \rangle$ for all $u \in V(G) \setminus N_G[v]$. This implies that the induced subgraph $\langle V(G) \setminus N_G[v] \rangle$of $V(G) \setminus N_G[v]$ is a clique in G. Further, since $S_H$is not a dominating set of H, if $v = x$ and $u \notin N_H[v]$, then $\{u\} \subseteq V(H) \setminus N_H[v]$ must be a dominating set of $\langle V(H) \setminus N_H[v] \rangle$ for all$u \in V(H) \setminus N_H[v]$. This implies that the induced subgraph $\langle V(H) \setminus N_H[v] \rangle$ of $V(H) \setminus N_H[v]$ is a clique in H. This satisfies (v)c).

Finally, if $\{z\} \subset S_G$ and$\{x\} \subset S_H$, then $|S_G| \geq 2$and $|S_H| \geq 2$. This satisfies (v)d).

For the converse, suppose that statement (i)a) is satisfied. Then by Lemma 2.5, S is a secure inverse dominating set of $G + H$.

Suppose that statement (i)b) is satisfied. Then by Lemma 2.6, S is a secure inverse dominating set of $G + H$.

Suppose that statement (ii)a) is satisfied. Then by Lemma 2.7, S is a secure inverse dominating set of $G + H$.

Suppose that statement (ii)b) is satisfied. Then by Lemma 2.8, S is a secure inverse dominating set of $G + H$.

Suppose that statement (iii)a) is satisfied. Then by Lemma 2.9, S is a secure inverse dominating set of$G + H$.

Suppose that statement (iii)b) is satisfied. Then by Lemma 2.10, S is a secure inverse dominating set of $G + H$.

Suppose that statement (iv) is satisfied. Then by Lemma 2.11, S is a secure inverse dominating set of $G + H$.

Suppose that statement (v)a) is satisfied. Then by Lemma 2.12, S is a secure inverse dominating set of $G + H$.

Suppose that statement (v)b) is satisfied. Then by Lemma 2.13, S is a secure inverse dominating set of $G + H$.

Suppose that statement (v)c) is satisfied. Then by Lemma 2.14, S is a secure inverse dominating set of $G + H$.

Suppose that statement (v)d) is satisfied. Then by Lemma 2.15, S is a secure inverse dominating set of $G + H$. This completes the proofs.□

The following result is a quick consequence of Theorem 2.16.

Corollary 2.17*Let G and H be connected non-complete graphs.*

$$\gamma_s^{(-1)}(G + H) = \begin{cases} 2, & if\ \gamma(G) = 1\ and\ \gamma_s(H) = 2 \\ 3, & if\ |S_G| = 2\ and\ \langle (V(G) \setminus S_G) \setminus N_G(S_G) \rangle\ is\ a\ clique\ in\ G. \end{cases}$$

Proof: Suppose that $\gamma(G) = 1$ and $\gamma_s(H) = 2$. Let D be a minimum dominating set of G with $|D| = 1$and S is a secure dominating set of H. Then by Theorem 2.16(i)b, S is a secure inverse dominating set of $G + H$. Thus, $\gamma_s^{(-1)}(G + H) \leq |S|$. Given that $\gamma_s(H) = 2$, let S be a minimum secure dominating set of H. Then

$$|S| = 2 = \gamma_s(H) \leq \gamma_s(G + H) \leq \gamma_s^{(-1)}(G + H) \leq |S|.$$

This implies that $\gamma_s^{(-1)}(G + H) = 2$.

Suppose that $|S_G| = 2$ and $\langle (V(G) \setminus S_G) \setminus N_G(S_G) \rangle$ is a clique in G. Let $S_H = \{x\}$. By Theorem 2.16(v)b, S is a secure inverse dominating set of $G + H$. Thus, $\gamma_s^{(-1)}(G + H) \leq |S|$. Let $S_G = \{v, z\}$. Then $S = \{v, z, x\}$, that is, $\gamma_s^{(-1)}(G + H) \leq |S| = 3$. Since, $\gamma(G) \neq 1$ and $\gamma(H) \neq 1$ by Theorem 2.16(v)b, it follows that $\gamma(G + H) = 2$ by Remark 2.4. Suppose that $S = \{v, x\}$ is a secure dominating set of $G + H$. Since $S_H = \{x\}$ is not a dominating set of H, there exists $w \in V(H) \setminus S_H$ such that $wx \notin E(H)$ and $(S \setminus \{v\}) \cup \{w\} = \{x, w\}$ is not a dominating set of H and of $G + H$. Thus, $\gamma_s(G + H) \neq 2$, that is, $\gamma_s(G + H) \geq 3$. By computation,

$$3 \leq \gamma_s(G + H) \leq \gamma_s^{(-1)}(G + H) \leq 3.$$

Hence, $\gamma_s^{-1}(G + H) = 3$.□

III. CONCLUSION

In this paper, a new parameter of domination in graphs was introduced - the secure inverse domination in graphs. The secure inverse domination in the join of two graphs were characterized. Moreover, the exact secure inverse domination number resulting from the join of two graphs were computed. This study will pave a way to new researches such as bounds and other binary operations of two connected graphs. Identifying the characterization of secure inverse domination in graphs of the corona, Cartesian product, and lexicographic product are promising extensions of this study. Finally, other parameters involving secure inverse domination in graphsmay also be explored.

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