Inverse Doubly Connected Domination in the Join and Cartesian Product of Two Graphs

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Abstract: Let G be a nontrivial connected graph. A dominating set $D \subseteq V(G)$ is called a doubly connected dominating set of G if both $\langle D \rangle$and $\langle V(G) \backslash D \rangle$ are connected.Let D be a minimum doubly connected dominating set of G. If $S \subseteq V(G) \backslash D$ is a doubly connected dominating set of G, then S is called an inverse doubly connected dominating set of G with respect to D. Furthermore, the inverse doubly connected domination number, denoted by $\gamma_{cc}^{-1}(G)$, is the minimum cardinality of an inverse doubly connected dominating set of G. An inverse doubly connected dominating set of cardinality$\gamma_{cc}^{-1}(G)$ is called γ_{cc}^{-1}-set. In this paper, we characterized the inverse doubly connected domination in the join and cartesian product of two graphs and give some important results.

Keywords: dominating set, doubly connected dominating set, inverse dominating set, inverse doubly connected dominating set

I. INTRODUCTION

The graphs G considered here are simple, finite, nontrivial, undirected and without isolated vertices.Domination in graph was introduced by Claude Berge in 1958 and Oystein Ore in 1962 [1]. Following an article [2] by Ernie Cockayne and Stephen Hedetniemi in 1977, the domination in graphs became an area of study by many researchers. A subset S of $V(G)$ is a *dominating set* of G if for every $v \in V(G) \backslash S$, there exists $x \in S$ such that $xv \in E(G)$, that is, $N[S] = V(G)$. The *domination number* $\gamma(G)$of G is the smallest cardinality of a dominating set of G. Some studies on domination in graphs were found in the papers [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24].

One variant of domination is the doubly connected domination in graphs. A dominating set $S \subseteq V(G)$ is called a *doubly connected dominating set* of G if both $\langle S \rangle$ and $\langle V(G) \backslash S \rangle$ are connected. The minimum cardinality of a doubly connected dominating set of G, denoted by $\gamma_{cc}(G)$, is called the *doubly connected domination number* of G. A doubly connected dominating set of cardinality$\gamma_{cc}(G)$ is called a γ_{cc} −set of G. Doubly connected domination in graphs is found in the papers [25, 26, 27].

The inverse domination in a graph was first found in the paper of Kulli [28] and studied in [29, 30, 31, 32, 33, 34, 35]. If D is a minimum dominating set in G, then a dominating set $S \subseteq V(G) \backslash D$ is called an inverse dominating set with respect to D. The inverse domination number, denoted by $\gamma^{-1}(G)$ of$G$is the order of an inverse dominating set with minimum cardinality.

In this paper, we introduced the study of inverse doubly connected domination in graphs. A dominating set $D \subseteq V(G)$ is called a doubly connected dominating set of G if both $\langle D \rangle$ and $\langle V(G) \backslash D \rangle$ are connected. Let D be a minimum doubly connected dominating set of G. If $S \subseteq V(G) \backslash D$ is a doubly connected dominating set of G, then S is called an inverse doubly connected dominating set of G with respect to D. Furthermore, the inverse doubly connected domination number, denoted by $\gamma_{cc}^{-1}(G)$ is the minimum cardinality of an inverse doubly connected dominating set of G. An inverse doubly connected dominating set of cardinality$\gamma_{cc}^{-1}(G)$ is called γ_{cc}^{-1}-set. In this paper, we characterized the inverse doubly connected domination in the join and cartesian product of two graphs and give some important results.

For the general terminology in graph theory, readers may refer to [36]. A *graph* Gis a pair $(V(G), E(G))$, where $V(G)$ is a finite nonempty set called the *vertex-set*of G and $E(G)$ is a set of unordered pairs $\{u, v\}$ (or simply uv) of distinct elements from $V(G)$ called the *edge-set* of G. The elements of $V(G)$ are called *vertices* and the cardinality $|V(G)|$ of $V(G)$ is the *order* of G. The elements of $E(G)$ are called *edges* and the cardinality $|E(G)|$ of $E(G)$ is the *size* of G. If $|V(G)| = 1$, then G is called a trivial graph. If $E(G) = \emptyset$, then G is called an empty graph. The *open neighborhood* of a vertex $v \in V(G)$ is the set $N_G(v) = \{u \in V(G): uv \in E(G)\}$. The elements of $N_G(v)$ are called *neighbors* of v.The *closed neighborhood* of $v \in V(G)$ is the set $N_G[v] = N_G(v) \cup \{v\}$. If $X \subseteq V(G)$, the *open neighborhood* of X in G is the set $N_G(X) = \bigcup_{v \in X} N_G(v)$. The *closed neighborhood* of X in G is the set $N_G[X] = \bigcup_{v \in X} N_G[v] = N_G(X) \cup X$. When no confusion arises, $N_G[x]$[resp. $N_G(x)$] will be denote by $N[x]$[resp. $N(x)$].

II. RESULTS

Definition 2.1 *A dominating set $D \subseteq V(G)$ is called a doubly connected dominating set of G if both $\langle D \rangle$ and $\langle V(G) \backslash D \rangle$ are connected. Let D be a minimum doubly connected dominating set of G. If $S \subseteq V(G) \backslash D$ is a doubly connected dominating set of G, then S is called an inverse doubly connected dominating set of G with respect to D. Furthermore, the inverse doubly connected domination number, denoted by $\gamma_{cc}^{-1}(G)$ is the minimum cardinality of an inverse doubly connected dominating set of* G. *An inverse doubly connected dominating set of cardinality$\gamma_{cc}^{-1}(G)$ is called γ_{cc}^{-1}-set.*

Definition 2.2 *The join of two graphs G and H is the graph $G + H$ with vertex-set $V(G + H) = V(G) \cup V(H)$ and edge-set $E(G + H) = E(G) \cup E(H) \cup \{uv: u \in V(G), v \in V(H)\}$.*

Remark 2.3 *Let G and H be a nontrivial connected graphs. Then $\gamma(G + H) \leq 2$.*

Remark 2.4 *Let G and H be nontrivial connected graphs. Then $V(G)$ and $V(H)$ are dominating sets of $G + H$.*

The following result is the characterization of an inverse doubly connected dominating set in the join of two graphs.

Theorem 2.5 *Let G and H be nontrivial connected graphs and $D \subset V(G + H)$ where $D \neq \emptyset$. Then a nonempty $S \subseteq V(G + H) \backslash D$ is an inverse doubly connected dominating set of $G + H$ with respect to D if and only if $|D| \leq 2$ and $\langle D \rangle$ is connected and one of the following is satisfied;*

(i) D is dominating set of G and $S \subseteq V(G) \backslash D$ is a connected dominating set of G, (or $S \subseteq V(H)$ is connected dominating set of H).

(ii) D is dominating set of H and $S \subseteq V(H) \backslash D$ is a connected dominating set of G, (or $S \subseteq V(G)$ is connected dominating set of G).

(iii) $D = D_G \cup D_H$ where $D_G \subset V(G)$ and $D_H \subset V(H)$ and S is a dominating set of $G + H$.

Proof: Suppose that a nonempty $S \subset V(G + H) \backslash D$ is an inverse doubly connected dominating set of $G + H$ with respect to D. Then D is a minimum dominating set of $G + H$ and S is a connected dominating set of $G + H$. By Remark 2.3, $|D| \leq 2$.

Case1. If $D \cap V(H) = \emptyset$ and $S \cap V(H) = \emptyset$ (or $S \cap V(G) = \emptyset$), then D is a dominating set of G and $S \subseteq V(G) \backslash D$ is a connected dominating set of G (or $S \subseteq V(H) \backslash D$ is a dominating set of H). This shows statement *(i)*.

Case2. If $D \cap V(G) = \emptyset$ and $S \cap V(G) = \emptyset$ (or $S \cap V(H) = \emptyset$), then D is a dominating set of H and $S \subseteq V(H) \backslash D$ is a connected dominating set of H (or $S \subseteq V(G) \backslash D$ is a dominating set of G). This shows statement *(ii)*.

Case3. If $D \cap V(G) \neq \emptyset$ and $D \cap V(H) \neq \emptyset$, then $V(G) \backslash D \neq \emptyset$ and $V(H) \backslash D \neq \emptyset$. Let $D_G = V(G) \backslash D$ and $D_H = V(H) \backslash D$. Thus,

$$\begin{aligned} D_G \cup D_H &= (V(G) \backslash D) \cup (V(H) \backslash D) \\ &= (V(G) \cap D) \cup (V(H) \cap D) \\ &= \big(V(G) \cup V(H)\big) \cap D \\ &= V(G + H) \cap D \\ &= D, \end{aligned}$$

that is $D = D_G \cup D_H$ where $D_G \subset V(G)$ and $D_H \subseteq V(H)$. Further, S is a dominating set of $G + H$. This shows statement *(iii)*.

For the converse, suppose that $|D| \leq 2$ and $\langle D \rangle$ is connected. If statement (i) is satisfied. Then D is a dominating set of G and hence a minimum connected dominating set of $G + H$ by Remark 2.3.

Case1. If $S \subseteq V(G)\backslash D$ is a connected dominating set of G, then $\langle V(G)\backslash D\rangle$ is connected (since S is a dominating set) and hence, D is a minimum doubly connected dominating set of G and $G+H$. Similarly, $D \subseteq V(G)\backslash S$ is a connected dominating set of G implies $\langle V(G)\backslash D\rangle$ is connected and hence, S is a doubly connected dominating set of G and $G+H$. Now, $S \subseteq V(G)\backslash D$ is an inverse dominating set of G with respect to D. This implies that $S \subseteq V(G+H)\backslash D$ is an inverse dominating set of $G+H$ with respect to D. Accordingly, a nonempty $S \subseteq V(G+H)\backslash D$ is an inverse doubly connected dominating set of $G+H$ with respect to D.

Case2. If $S \subseteq V(H)\backslash D$ is a connected dominating set of H, then $\langle V(G+H)\backslash D\rangle$ is clearly connected and hence, D is a minimum doubly connected dominating set of $G+H$. Similarly, $D \subseteq V(G+H)\backslash S$ is a connected dominating set of $G+H$ implies $\langle V(G+H)\backslash S\rangle$ is connected and hence, S is a doubly connected dominating set of $G+H$. Now, $S \subseteq V(G+H)\backslash D$ is an inverse dominating set of $G+H$ with respect to D. Accordingly, a nonempty $S \subseteq V(G+H)\backslash D$ is an inverse doubly connected dominating set of $G+H$ with respect to D.

If statement (*ii*) is satisfied. Then D is a dominating set of $H$and hence a minimum connected dominating set of $G+H$ by Remark 2.3.

Case1. If $S \subseteq V(H)\backslash D$ is a connected dominating set of H, then $\langle V(H)\backslash D\rangle$ is connected (since S is a dominating set) and hence, D is a minimum doubly connected dominating set of H and $G+H$. Similarly, $D \subseteq V(H)\backslash S$ is a connected dominating set of H implies $\langle V(H)\backslash S\rangle$ is connected and hence, S is a doubly connected dominating set of H and $G+H$. Now, $S \subseteq V(H)\backslash D$ is an inverse dominating set of H with respect to D. This implies that $S \subseteq V(G+H)\backslash D$ is an inverse dominating set of $G+H$ with respect to D. Accordingly, a nonempty $S \subseteq V(G+H)\backslash D$ is an inverse doubly connected dominating set of $G+H$ with respect to D.

Case2. If $S \subseteq V(G)$ is a connected dominating set of G, then $\langle V(G+H)\backslash D\rangle$ is clearly connected and hence, D is a minimum doubly connected dominating set of $G+H$. Similarly, $D \subset V(G+H)\backslash S$ is a connected dominating set of $G+H$ implies $\langle V(G+H)\backslash S\rangle$ is connected and hence, S is a doubly connected dominating set of $G+H$. Now, $S \subseteq V(G+H)\backslash D$ is an inverse dominating set of $G+H$ with respect to D. Accordingly, a nonempty $S \subseteq V(G+H)\backslash D$ is an inverse doubly connected dominating set of $G+H$ with respect to D.

If statement (*iii*) is satisfied. Then $D = D_G \cup D_H$ where $D_G \subset V(G)$ and $D_H \subseteq V(H)$ and S is a dominating set of $G+H$. Note that $D \neq \emptyset$. Consider the following cases.

Case1. If $D_H = \emptyset$, then $D = D_G$. Since $|D| \leq 2$ and $\langle D\rangle$ is connected, D is a dominating set of G and hence a minimum connected dominating set $G+H$ by the Remark 2.3. Given that S is a dominating set of $G+H$, if S is a dominating set of G (or H), then by similar proof of statement (*i*), a nonempty $S \subseteq V(G+H)\backslash D$ is an inverse doubly connected dominating set of $G+H$ with respect to D. Otherwise, $S \subseteq V(G+H)\backslash D = (V(G)\backslash D) \cup V(H)$, that is $\langle V(G+H)\backslash D\rangle$ is connected and S is a connected dominating set of $G+H$. Hence, D is a minimum doubly connected dominating set of $G+H$ and $V(G+H)\backslash S \supseteq D$ implies that $\langle V(G+H)\backslash S\rangle$ is connected (since D is a dominating set). Thus, S is a doubly connected dominating set of $G+H$. Since $S \subseteq V(G+H)\backslash D$, it follows that S is an inverse dominating set of $G+H$ with respect to D. Accordingly, a nonempty $S \subseteq V(G+H)\backslash D$ is an inverse doubly connected dominating set of $G+H$ with respect to D.

Case2. If $D_G = \emptyset$, then $D = D_H$. Since $|D| \leq 2$ and $\langle D\rangle$ is connected, D is a dominating set of H and hence a minimum connected dominating set $G+H$ by the Remark 2.3. Given that S is a dominating set of $G+H$, if S is a dominating set of H (or G), then by similar proof of statement (*ii*), a nonempty $S \subseteq V(G+H)\backslash D$ is an inverse doubly connected dominating set of $G+H$ with respect to D. Otherwise, $S \subseteq V(G+H)\backslash D = V(G) \cup (V(H)\backslash D)$, that is $\langle V(G+H)\backslash D\rangle$ is connected and S is a connected dominating set of $G+H$. Hence, D is a minimum doubly connected dominating set of $G+H$ and $V(G+H)\backslash S \supseteq D$ implies that $\langle V(G+H)\backslash S\rangle$ is connected (since D is a dominating set). Thus, S is a doubly connected dominating set of $G+H$. Since $S \subseteq V(G+H)\backslash D$, it follows that S is an inverse dominating set of $G+H$ with respect to D. Accordingly, a nonempty $S \subseteq V(G+H)\backslash D$ is an inverse doubly connected dominating set of $G+H$ with respect to D.

Case3. If $D_G \neq \emptyset$ and $D_H \neq \emptyset$, then $D = D_G \cup D_H$. Given that S is a dominating set of $G+H$, if $\gamma(G) = 1$, then by a similar proof to statement (*i*) a nonempty set of $S \subseteq V(G+H)\backslash D$ is an inverse doubly connected dominating set of $G+H$ with respect to D. If $\gamma(H) = 1$, then by a similar proof to statement (*ii*) a nonempty set of $S \subseteq V(G+H)\backslash D$ is an inverse doubly connected dominating set of $G+H$ with respect to D. Finally, if

$\gamma(G) \neq 1$ and $\gamma(G) \neq 1$, $|D| = 2$ (since $|D| \neq 1$) and $\langle D \rangle$ is connected, then let $D_G = \{x\}$ and $D_H = \{y\}$. This implies that D is a minimum connected dominating set of $G + H$ (because $\gamma(G + H) \neq 1$). Clearly, $\langle V(G + H)\backslash D \rangle$ is connected (since G and H are connected non trivial graphs), that is, D is a minimum doubly connected dominating set of $G + H$. Now, given that S is a dominating set of $G + H$, if S is a dominating set of G then by similar proof of statement (*i*) a nonempty $S \subseteq V(G + H)\backslash D$ is an inverse doubly connected dominating set of $G + H$ with respect to D.If S is a dominating set of H, then by similar proof of statement (*ii*) a nonempty $S \subseteq V(G + H)\backslash D$ is an inverse doubly connected dominating set of $G + H$ with respect to D. If S is not a dominating set of G and S is a dominating set of H, then let $S = S_G \cup S_H$ such that $S_G \subseteq V(G)\backslash\{x\}$ and $S_H \subseteq V(H)\backslash\{y\}$. Then $\langle S \rangle$ is connected and $\langle V(G + H)\backslash S \rangle$ is connected. Hence, S is a doubly connected dominating set of $G + H$. Since $S \subseteq V(G + H)\backslash D$, it follows that S is an inverse doubly connected dominating set of $G + H$ with respect to D. ∎

The following result is an immediate consequence of Theorem 2.5.

Corollary 2.6*Let G and H be nontrivial connected graphs. Then*

$$\gamma_{cc}^{-1}(G + H) = \begin{cases} 1 \; if \; \gamma(G) = 1 \; and \; \gamma(H) = 1, \\ 2 \; if \; \gamma(G) \neq 1 \; and \; \gamma(H) \neq 1. \end{cases}$$

Proof. Suppose that $\gamma(G) = 1$ and $\gamma(H) = 1$. Then let $D = D_G = \{x\}$ be a dominating set of G. Then $D = D_G \cup D_H$ with $D_G = \{x\}$ and $D_H = \emptyset$. Let $S = \{y\}$ be a dominating set of H and hence $G + H$. In view of Theorem 2.5(*iii*), S is an inverse doubly connected dominating set of $G + H$. Thus, $\gamma_{cc}^{-1}(G + H) \leq |S| = 1$. Since $1 \leq \gamma_{cc}^{-1}(G + H) \leq |S| = 1$, it follows that $\gamma_{cc}^{-1}(G + H) = 1$.

Suppose that $\gamma(G) \neq 1$ and $\gamma(H) \neq 1$. Let $D_G = \{x\} \subset V(G)$ and $D_H = \{y\} \subset V(H)$ such that $D = D_G \cup D_H$. Let $S = \{u, v\}$ with $u \in V(G)\backslash D_G$ and $v \in V(H)\backslash D_H$. Then S is a dominating set of $G + H$. By Theorem 2.5(*iii*), S is an inverse doubly connected dominating set of $G + H$. Thus, $\gamma_{cc}^{-1}(G + H) \leq |S| = 2$. Since $\gamma(G + H) \neq 1$ is immediate, it follows that $\gamma_{cc}^{-1}(G + H) \geq 2$ is clear. Thus, $2 \leq \gamma_{cc}^{-1}(G + H) \leq |S| = 2$. Therefore, $\gamma_{cc}^{-1}(G + H) = 2$. ∎

Definition 2.7 *The Cartesian product $G \square H$ of two graphs G and H is the graph with $V(G \square H) = V(G) \times V(H)$ and $(u, u')(v, v') \in E(G \square H)$ if and only if either $uv \in E(G)$ and $u' = v'$ or $u = v$ and $u'v' \in E(H)$.*

Theorem 2.8 *Let $G = P_n = [v_1, v_2, \cdots, v_n]$ and $H = C_4 = [u_1, u_2, u_3, u_4]$ where $n \geq 1$. Then $S \subseteq V(G \square H)\backslash D$ is an inverse doubly connected dominating set of $G \square H$ with respect to a minimum doubly connected dominating set D of $G \square H$, if and only if $D = \{(v_k, u_1), (v_k, u_2): k = 1, 2, \cdots, n\}$ and $S = \{(v_k, u_3), (v_k, u_4): k = 1, 2, \cdots, n\}$.*

Proof. Let $G = P_n = [v_1, v_2, \cdots, v_n]$andH $= C_4 = [u_1, u_2, u_3, u_4]$where $n \geq 1$. Suppose that$S \subseteq V(G \square H)\backslash D$is an inverse doubly connected dominating set of $G \square H$ with respect to a minimum connected dominating set $D$of$G \square H$. Let $D_H = \{u_1, u_2\}$. Then $\langle D_H \rangle$ and $\langle V(H)\backslash D_H \rangle = \langle \{u_3, u_4\} \rangle$ are both connected dominating set of H, that is, D_H is a minimum doubly connected dominating set of H. Thus,

$$D = V(G) \times D_H = \{v_k : k = 1, 2, \cdots, n\} \times \{u_1, u_2\} = \{(v_k, u_1), (v_k, u_2): k = 1, 2, \cdots, n\}.$$

and

$$\begin{aligned} S &= V(G \square H)\backslash D \\ &= \big(V(G) \times V(H)\big)(V(G) \times D_H) \\ &= V(G) \times (V(H)\backslash D_H) \\ &= \{v_k : k = 1, 2, \cdots, n\} \times \{u_3, u_4\} \\ &= \{(v_k, u_3), (v_k, u_4): k = 1, 2, \cdots, n\}. \end{aligned}$$

Forthe converse, suppose that $D = \{(v_k, u_1), (v_k, u_2): k = 1, 2, \cdots, n\}andS = \{(v_k, u_3), (v_k, u_4): k = 1, 2, \cdots, n\}$. Then

$$D = \{(v_k, u_1), (v_k, u_2): k = 1, 2, \cdots, n\} = \{v_k : k = 1, 2, \cdots, n\} \times \{u_1, u_2\} = V(G) \times D_H.$$

and

$$S = \{(v_k, u_3), (v_k, u_4): k = 1, 2, \cdots, n\} = \{v_k : k = 1, 2, \cdots, n\} \times \{u_3, u_4\} = V(G) \times (V(H)\backslash D_H).$$

Since, $D_H = \{u_1, u_2\}$ is a minimum connected dominating set of H, it follows that $D = V(H) \times D_H$ is a minimum connected dominating set of $G \square H$. Further, $\langle D_H \rangle$ and $\langle V(H)\backslash D_H \rangle = \langle \{u_3, u_4\} \rangle$ are both connected in H. This implies that

$\langle D_H \rangle = \langle V(G) \times D_H \rangle$ is connected in $G \square H$

and

$\langle V(G \square H)\backslash D \rangle = \langle (V(G) \times V(H))\backslash(V(G) \times D_H) \rangle = \langle V(G) \times (V(H)\backslash D_H) \rangle$ is connected in $G \square H$

Thus, D is a minimum doubly connected dominating set of $G\square H$.

Since $V(H)\backslash D_H = \{u_3, u_4\}$ is connected dominating set of H, it follows that $S = V(G) \times (V(H)\backslash D_H)$ is a connected dominating set of $G\square H$. Further, $\langle D_H \rangle$ and $\langle V(H)\backslash D_H \rangle = \langle \{u_3, u_4\} \rangle$ are both connected in H. This implies that
$\langle S \rangle = \langle V(G) \times (V(H)\backslash D_H) \rangle$ is connected in $G\square H$
and

$$\langle V(G\square H)\backslash S \rangle = \langle (V(G) \times V(H))\backslash (V(G) \times (V(H)\backslash D_H)) \rangle = \langle V(G) \times D_H \rangle \text{ is connected in } G\square H$$

Thus, S is a doubly connected dominating set of $G\square H$. Since $S = V(G\square H)\backslash D$, it follows that S is an inverse doubly connected dominating set with respect to D. Accordingly, $S = V(G\square H)\backslash D$ is an inverse doubly connected dominating set of $G\square H$ with respect to a minimum doubly connected dominating set D of $G\square H$. ■

The following result, is an immediate consequence of Theorem 2.8.

Corollary 2.9 *Let* $G = P_n = [v_1,\ v_2,\ \cdots,\ v_n]$ *and* $H = C_4 = [u_1,\ u_2,\ u_3,\ u_4]$ *where* $n \geq 1$. *Then* $\gamma_{cc}^{-1}(G\square H) = 2n$.

Proof: Suppose that $D_H = \{u_1, u_2\}$. Then $V(H)\backslash D_H = \{u_3, u_4\}$. Let $D = V(G) \times D_H$ and $S = V(G) \times (V(H)\backslash D_H)$. Then

$$D = V(G) \times D_H = \{v_k : k = 1, 2,\ \cdots, n\} \times \{u_1, u_2\} = \{(v_k, u_1), (v_k, u_2) : k = 1, 2,\ \cdots, n\}.$$

and

$$S = V(G) \times (V(H)\backslash D_H) = \{v_k : k = 1, 2,\ \cdots, n\} \times \{u_3, u_4\} = \{(v_k, u_3), (v_k, u_4) : k = 1, 2,\ \cdots, n\}.$$

By Theorem 2.8, S is an inverse doubly connected dominating set of $G\square H$ with respect to a minimum doubly connected dominating set of D of $G\square H$. Thus, $\gamma_{cc}^{-1}(G\square H) \leq |S| = |\{(v_k, u_3), (v_k, u_4) : k = 1, 2,\ \cdots, n\}| = 2n$. Since $D=VG\times DH$ is a minimum doubly connected dominating set of $G\square H$ and

$$\begin{aligned}\gamma_{cc}(G\square H) = |D| &= |\,V(G) \times D_H| \\ &= |\{v_k : k = 1, 2,\ \cdots, n\} \times \{u_1, u_2\}| \\ &= |\{(v_k, u_1), (v_k, u_2) : k = 1, 2,\ \cdots, n\}| \\ &= 2n,\end{aligned}$$

it follows that $2n = \gamma_{cc}(G\square H) \leq \gamma_{cc}^{-1}(G\square H) = 2n$. Hence, $\gamma_{cc}^{-1}(G\square H) = 2n$. ■

III. CONCLUSION

In this paper, we introduced a new parameter of domination of graphs - the inverse doubly connected domination in graphs. The inverse doubly connected domination in the join and cartesian product of two graphs were characterized. The exact inverse doubly connected domination number resulting from the join and cartesian product of two graphs were computed. This study will pave a way to new researches such bounds and other binary operations of two connected graphs. Other parameters involving inverse doubly connected domination in graphs may also be explored. Finally, the characterization of an inverse doubly connected domination in graphs of the lexicographic product, and its bounds are promising extension of this study.

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