International Journal of Latest Engineering Research and Applications (IJLERA) ISSN: 2455-7137

Volume – 09, Issue – 01, January 2024, PP – 43-49

Disjoint Perfect Domination in Join and Corona of Two Graphs

Ma. Elizabeth N. Diapo[1], Enrico L. Enriquez[2]

[1,2]*Department of Computer, Information Science and Mathematics*

School of Arts and Sciences

University of San Carlos, 6000 Cebu City, Philippines

Abstract: Let G bea connected simple graph. A dominating set $S \subseteq V(G)$ is called a perfect dominating set of G if every $u \in V(G)\backslash S$is dominated by exactly one element of S. Let D be a minimum perfect dominating set of G. A perfect dominating set $S \subset (V(G)\backslash D)$ is called an inverse perfect dominating set of G with respect to D. A disjoint perfect dominating set of G is the set $C = D \cup S \subseteq V(G)$. Furthermore, the disjoint perfect domination number, denoted by, $\gamma_p\gamma_p(G)$, is the minimum cardinality of a disjoint perfect dominating set of G. A disjoint perfect dominating set of cardinality, $\gamma_p\gamma_p(G)$, is called $\gamma_p\gamma_p$-set. In this paper, we give the characterization of the disjoint perfect dominating sets of the join and corona of two graphs. We also determine the disjoint perfect domination number of the join and corona of two graphs and give some important results.

Keywords: dominating set, perfect dominating set, inverse perfect dominating set, disjoint perfect dominating set, join of two graphs, and corona of two graphs

.

I. INTRODUCTION

Domination in graphs was introduced by Claude Berge in 1958 and Oystein Ore in 1962 [1] and studied in the papers [2, 3, 4]. One type of domination in graphs is the perfect domination in graphs. This was introduced by Cockayne et.al [5] in the paper, *perfect domination in graphs* and studied further in [6, 7, 8, 9]. The *inverse domination in a graph* was found in [10, 11, 12, 13, 14, 15, 16]. Further, the papers on *disjoint domination in graphs* can be read in [17, 18, 19, 20]. For the general concepts, readers may refer to [22].

A *graphG* is a pair $(V(G), E(G))$where $V(G)$is a finite nonempty set called the *vertex-set* of *G* and $E(G)$ is a set of unordered pairs u, v (or simply uv) of distinct elements from $V(G)$ called the *edge-set* of *G*. The elements of $V(G)$ are called *vertices* and the cardinality $|V(G)|$ of $V(G)$ is the *order* of *G*. The elements of $E(G)$ are called *edges* and the cardinality $|E(G)|$ of $E(G)$ is the *size* of *G*. If $|V(G)| = 1$, then *G* is called a *trivial graph*. If $E(G) = \emptyset$, then *G* is called an *empty graph*. The *open neighborhood* of a vertex $v \in V(G)$ is the set $N_G(v) = \{u \in V(G): uv \in E(G)\}$. The elements of $N_G(v)$ are called *neighbors* of *v*. The *closed neighborhood* of $v \in V(G)$ is the set, $N_G[v] = N_G(v) \cup \{v\}$. If $X \subseteq V(G)$, the *open neighborhood* of *X* in *G* is the set$N_G(X) = \bigcup_{v \in X} N_G(v)$. The *closed neighborhood* of *X* in *G* is the set $N_G[X] = \bigcup_{v \in X} N_G[v] = N_G(X) \cup X$. When no confusion arises, $N_G[x]$[resp.$N_G(x)$] will be denoted by $N[x]$ [resp.$N(x)$].

Let *G* be a simple connected graph. A subset *S* of a vertex set $V(G)$ is a *dominating set* of *G* if for every vertex $v \in V(G)\backslash S$, there exists a vertex $x \in S$ such that xv is an edge of G. The *domination number*$\gamma(G)$ of *G* is the smallest cardinality of a dominating set *S* of *G*.

A dominating set $S \subseteq V(G)$ is called a *perfect dominating set* of *G* if each $u \in V(G)\backslash S$ is dominated by exactly one element of *S*. The *perfect domination number* of *G*, denoted by $\gamma_p(G)$ is the minimum cardinality of a perfect dominating set of *G*. Let *D* be a minimum dominating set in *G*. The dominating set $S \subseteq V(G)\backslash D$ is called an *inverse dominating set* with respect to *D*. The minimum cardinality of inverse dominating set is called an *inverse domination number* of *G* and is denoted by, $\gamma^{-1}(G)$. An inverse dominating set of cardinality $\gamma^{-1}(G)$ is called γ^{-1}*-set* of G.

Salve and Enriquez [23] define the inverse perfect domination in graphs. Let *D* be a minimum perfect dominating set of *G*. A perfect dominating set $S \subseteq V(G)\backslash D$ is called an *inverse perfect dominating set*of*G* with respect to *D*. The *inverse perfect domination number* of *G* denoted by $\gamma_P^{-1}(G)$ is the minimum cardinality of an inverse perfect dominating set of *G*. An inverse perfect dominating set of cardinality $\gamma_P^{-1}(G)$ is called γ_P^{-1}*-set.*

In this paper, the researchers extend the concept of inverse perfect domination in graphs by introducing the disjoint perfect domination in graphs. Let *D* be a minimum perfect dominating set of *G* and $S \subseteq (V(G)\backslash D)$ is an inverse perfect dominating set of *G* with respect to *D*. A *disjoint perfect dominating set* of *G* is the set $C = D \cup S \subseteq V(G)$. Furthermore, the *disjoint perfect domination number*, denoted by $\gamma_p\gamma_p(G)$, is the minimum cardinality of a disjoint perfect dominating set of G. A disjoint perfect dominating set of cardinality $\gamma_p\gamma_p(G)$ is called $\gamma_p\gamma_p(G) - set$. The researchers give the characterization of the disjoint perfect domination in the join and corona two graphs and give some important results. Unless otherwise stated, all subsets of the vertex sets in this paper are assumed to be nonempty.

II. RESULTS

Since the $\gamma_P^{-1}(G)$ does not always exist in a connected nontrivial graph G by Salve et.al. [23], the researchers introduce $\mathcal{DP}(G)$ as a family of all graphs with inverse perfect dominating set and disjoint perfect dominating set. Thus, for the purpose of this study, it is assumed that all connected nontrivial graphs considered belong to the family$\mathcal{DP}(G)$.

We need the following remarks for the characterizations of disjoint perfect domination in the join and corona of graphs.

Remark 2.1*Let G be a connected nontrivial graph. Then* $\gamma(G) = \gamma_P(G) = \gamma_P^{-1}(G) = 1$.

Theorem 2.2 [23] *Let G be a connected graph of order* $n \geq 2$. *Then,* $\gamma_p^{-1}(G) = 1$ *if and only if* $G = K_1 + H$ *where* $\gamma(H) = 1$.

Definition 2.3*The join of two graphs G and H is the graph* $G + H$ *with vertex-set* $V(G + H) = V(G) \cup V(H)$ *and edge-set*

$$E(G + H) = E(G) \cup E(H) \cup \{uv\colon u \in V(G),\ v \in V(H)\}.$$

Remark 2.4*Let G and H be any graphs. Then,* $\gamma_p\gamma_p(G + H) \neq 1$.

Lemma 2.5*Let G and H be nontrivial graphs,* $D \subset V(G + H)$*and* $S \subset V(G + H)\backslash D$. *If* $D = \{x\}$ *and* $S = \{y\}$ *are dominating sets of G, then a nonempty set*$C = D \cup S$ *is a disjoint perfect dominating set of* $G + H$.

Proof: Suppose that $D = \{x\}$ and $S = \{y\}$ are dominating sets of G and hence of $G + H$, then D and S are perfect dominating sets of $G + H$. Now, $S \subset V(G + H)\backslash D$ implies that S is an inverse perfect dominating set of $G + H$ with respect to D. Thus, a nonempty set $C = D \cup S$ is a disjoint perfect dominating set of $G + H$. ■

Lemma 2.6*Let G and H be nontrivial graphs,* $D \subset V(G + H)$*and* $S \subseteq V(G + H)\backslash D$. *If* $D = \{x\}$ *and* $S = \{y\}$ *are dominating sets of H, then a nonempty set* $C = D \cup S$ *is a disjoint perfect dominating set of* $G + H$.

Proof : Suppose that $D = \{x\}$ and $S = \{y\}$ are dominating sets of H and hence of $G + H$, then D and S are perfect dominating sets of $G + H$. Now, $S \subset V(G + H)\backslash D$ implies that S is an inverse perfect dominating set of $G + H$ with respect to D. Thus, a nonempty set $C = D \cup S$ is a disjoint perfect dominating set of $G + H$. ■

Lemma 2.7*Let G and H be nontrivial graphs,* $D \subset V(G + H)$*and* $S \subseteq V(G + H)\backslash D$. *If* $D = \{x\}$ *and* $S = \{y\}$ *are dominating sets of G and H respectively, then a nonempty set* $C = D \cup S$ *is a disjoint perfect dominating set of* $G + H$.

Proof: Suppose that $D = \{x\}$ and $S = \{y\}$ are dominating sets of G and H respectively and hence of $G + H$, it follows that D and S are perfect dominating sets of $G + H$. Now, $S \subseteq V(G + H)\backslash D$ implies that S is an inverse perfect dominating set of $G + H$ with respect to D. Thus, a nonempty set $C = D \cup S$ is a disjoint perfect dominating set of $G + H$. ■

Lemma 2.8*Let G and H be nontrivial graphs,* $D \subset V(G + H)$*and* $S \subseteq V(G + H)\backslash D$. *If* $D = \{x\}$ *and* $S = \{y\}$ *are dominating sets of H and G respectively, then a nonempty set* $C = D \cup S$ *is a disjoint perfect dominating set of* $G + H$.

Proof: Suppose that $D = \{x\}$ and $S = \{y\}$ are dominating sets of H and G respectively and hence of $G + H$, it follows that D and S are perfect dominating sets of $G + H$. Now, $S \subseteq V(G + H)\backslash D$ implies that S is an inverse perfect dominating set of $G + H$ with respect to D. Thus, a nonempty set $C = D \cup S$ is a disjoint perfect dominating set of $G + H$. ■

Lemma 2.9*Let G and H be nontrivial graphs,* $D \subset V(G + H)$*and* $S \subseteq V(G + H)\backslash D$. *If* $D = \{x, v\}$ *and* $S = \{y, u\}$, *where x and y are distinct isolated vertices of G, v and u are distinct isolated vertices of H, then a nonempty set* $C = D \cup S$ *is a disjoint perfect dominating set of* $G + H$.

Proof: Suppose that $D = \{x, v\}$ and $S = \{y, u\}$, where x and y are distinct isolated vertices of G, v and u are distinct isolated vertices of H. Clearly $D = \{x, v\}$ is a dominating set of $G + H$. Since $x \in D$ is an isolated vertex in G, $xw \in E(G + H)$ for all $w \in V(H)$. Since $v \in D$ is an isolated vertex in H, $vz \in E(G + H)$ for all $z \in V(G)$. Thus, each vertex in $V(G + H)\backslash D$ is dominated by exactly one element in D. Hence, D is a perfect dominating set of $G + H$. Similarly, $S = \{y, u\}$ is a perfect dominating set of $G + H$. Now, $S \subseteq V(G + H)\backslash D$ implies that S is an inverse perfect dominating set of $G + H$ with respect to D. Thus, a nonempty set $C = D \cup S$ is a disjoint perfect dominating set of $G + H$. ■

The next result presents a characterization of disjoint perfect dominating sets in the join of two connected graphs and gives the corresponding disjoint perfect domination number.

Theorem 2.10*Let G and H be nontrivial graphs, $D \subset V(G+H)$ and $S \subseteq V(G+H)\backslash D$.Then, a nonempty set $C = D \cup S$ is a disjoint perfect dominating set of$G+H$ if and only one of the following is satisfied.*

(i) $D = \{x\}$ and $S = \{y\}$ are dominating sets of G, or
$D = \{x\}$and$S = \{y\}$are dominating sets of H.

(ii) $D = \{x\}$ and $S = \{y\}$ are dominating sets of G and H respectively, or
$D = \{x\}$and$S = \{y\}$ are dominating sets of H and G respectively.

(iii) $D = \{x, v\}$and $S = \{y, u\}$, where x and y are distinct isolated vertices ofG, v and u are distinct isolated vertices of H.

Proof: Suppose that a nonempty set $C = D \cup S$ is a disjoint perfect dominating set of $G+H$ where$D \subset V(G+H)$and $S \subseteq V(G+H)\backslash D$. Then D is a minimum perfect dominating set of $G+H$ and S is an inverse perfect dominating set with respect to D. Consider the following cases.

Case1. Suppose that $D \cap V(H) = \emptyset$ and$S \cap V(H) = \emptyset$. Then, $D, S \subset V(G)$, that is, D and S are dominating sets of G and perfect dominating sets of $G+H$. If $|D| \neq 1$, then $|D| \geq 2$, that is, for every $u \in V(H)$, $uv \in E(G+H)$ for all $v \in S$. This contradicts the definition of a perfect dominating set. Hence, $|D| = 1$ and let $D = \{x\}$. Similarly, if$|S| \neq 1$, then S is not a perfect dominating set of $G+H$. Hence, $|S| = 1$ and let $D = \{x\}$. Therefore, $D = \{x\}$ and $S = \{y\}$ and are dominating sets of G, showing statement (*i*).

Case2. Suppose that $D \cap V(G) = \emptyset$ and $S \cap V(G) = \emptyset$. Then, $D, S \subset V(H)$, that is, D and S are dominating sets of H and perfect dominating sets of $G+H$. If $|D| \neq 1$,then $|D| \geq 2$, that is, for every $u \in V(G)$, $uv \in E(G+H)$ for all $v \in S$. This contradicts the definition of a perfect dominating set. Hence, |D| = 1 and let D = $\{x\}$. Similarly, if |S| ≠ 1, then S is not a perfect dominating set of $G+H$. Hence, $|S| = 1$ and let $D = \{x\}$. Therefore, $D = \{x\}$ and $S = \{y\}$ are dominating sets of H, showing statement (*i*).

Case3. Suppose that $D \cap V(H) = \emptyset$ and $S \cap V(G) = \emptyset$. Then $D \subset V(G)$ and $S \subset V(H)$, that is, D and S are dominating sets of G and Hrespectively. Similarly, if $|D| \neq 1$and $|S| \neq 1$, then D and S are not perfect dominating sets, contrary to the assumption. Thus, $|D| = 1$and $|S| = 1$. Let $D = \{x\}$ and $S = \{y\}$. Then $D = \{x\}$ and $S = \{y\}$ are dominating sets of G and H respectively, showing statement (*ii*).

Case4. Suppose that $D \cap V(G) = \emptyset$ and $S \cap V(H) = \emptyset$. Then $D \subset V(H)$ and $S \subset V(G)$, that is, D and S are dominating sets of H and G respectively. Similarly, if $|D| \neq 1$and $|S| \neq 1$, then D and S are not perfect dominating sets, contrary to the assumption. Thus, $|D| = 1$and $|S| = 1$. Let $D = \{x\}$ and $S = \{y\}$. Then $D = \{x\}$ and $S = \{y\}$ are dominating sets of H and Grespectively, showing statement (*ii*).

Case5. Suppose that $D \cap V(G) \neq \emptyset$ and $D \cap V(H) \neq \emptyset$ and $S \cap V(G) \neq \emptyset$ and $S \cap V(H) \neq \emptyset$. Let $D_G = D \cap V(G)$and$D_H = D \cap V(H)$ and $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$. This implies that

$$\begin{aligned} D_G \cup D_H &= \big(D \cap V(G)\big) \cup \big(D \cap V(H)\big) \\ &= D \cap \big(V(G) \cup V(H)\big) \\ &= D \cap V(G+H) \\ &= D, \end{aligned}$$

and

$$\begin{aligned} S_G \cup S_H &= \big(S \cap V(G)\big) \cup \big(S \cap V(H)\big) \\ &= S \cap (V(G) \cup V(H) \\ &= S \cap V(G+H) \\ &= S. \end{aligned}$$

Thus $D = D_G \cup D_H$ and $S = S_G \cup S_H$.SinceD_G, D_H, S_G, and S_H are nonempty sets, it follows that $|D| \geq 2$ and $|S| \geq 2$. Since D and S are perfect dominating sets, $|D| \leq 2$ and $|S| \leq 2$ by Remark 2.4. Thus, $2 \leq |D| \leq 2$and $2 \leq |S| \leq 2$, that is, $|D| = 2$ and $|S| = 2$. Let $D = \{x, v\}$ and $S = \{y, u\}$, where x and y are distinct vertices of G, v and u are distinct vertices of H. If x is not an isolated vertex of G, then there exists $y \in V(G)$ such that $xy, vy \in E(G+H)$. Since $D = \{x, v\}$, it follows that y is dominated by two elements of D contrary to our assumption that D is a perfect dominating set of $G+H$. Thus, x must be an isolated vertex of G. Similarly, v is an isolated vertex of H. Further, since $S = \{y, u\}$ is a perfect dominating set of $G+H$, y and $u$must be isolated vertices of G and H respectively. Therefore,$D = \{x, v\}$ and $S = \{y, u\}$, where x and y are distinct isolated vertices of G, v and u are distinct isolated vertices of H. This shows statement (*iii*).

For the converse, suppose that statement (*i*) is satisfied, that is, $D = \{x\}$ and $S = \{y\}$ are dominating sets of G. Then by Lemma 2.5, a nonempty set $C = D \cup S$is a disjoint perfect dominating set of $G + H$.

Suppose that statement (*i*) is satisfied, that is, $D = \{x\}$ and $S = \{y\}$ are dominating sets of G. Then by Lemma 2.6, a nonempty set $C = D \cup S$ is a disjoint perfect dominating set of$G + H$.

Suppose that statement (*ii*) is satisfied, that is, $D = \{x\}$ and $S = \{y\}$ are dominating sets of G and H respectively. Then by Lemma 2.7, a nonempty set $C = D \cup S$is a disjoint perfect dominating set of $G + H$.

Suppose that statement (*ii*) is satisfied, that is, $D = \{x\}$ and $S = \{y\}$ are dominating sets of H and G respectively. Then by Lemma 2.8, a nonempty set $C = D \cup S$is a disjoint perfect dominating set of $G + H$.

Suppose that statement (*iii*) is satisfied. Then by Lemma 2.9, a nonempty set $C = D \cup S$is a disjoint perfect dominating set of $G + H$. ∎

Corollary 2.11*Let G and H be nontrivial graphs.*

$$\gamma_P\gamma_P(G + H) = \begin{cases} 2, & if\ \gamma(G) = 1\ and\ \gamma(H) = 1\ or\ G + H = K_2 + H \\ 4, & if\ each\ G\ and\ H\ has\ at\ least\ two\ isolated\ vertices. \end{cases}$$

Proof: Suppose that $\gamma(G) = 1$ and $\gamma(H) = 1$. Let $D = \{x\}$ and $S = \{y\}$ be dominating sets of G and H, respectively. Then by Theorem 2.10, $C = D \cup S$ is a disjoint perfect dominating set of $G + H$. Thus, $\gamma_P\gamma_P(G + H \leq C = 2$. Since, $\gamma P\gamma P(G+H) \neq 1$, by Remark 2.4, it follows that $\gamma P\gamma PG+H \geq 2$. Thus,

$$2 \leq \gamma_p\gamma_p(G + H) \leq |C| = 2$$

implies that$\gamma_p\gamma_p(G + H) = 2$.

Suppose that $G + H = K_2 + H$. Let $V(G) = V(K_2) = \{x, y\}$ where $D = \{x\}\ and\ S = \{y\}$. Then D and S are dominating sets of G. By Theorem 2.10, $C = D \cup S$ is a disjoint perfect dominating set of $G + H$. Thus, $\gamma_P\gamma_P(G + H) \leq |C| = 2$. Similarly, by Remark 2.4, it follows that $\gamma_P\gamma_P(G + H) = 2$.

Suppose that each *G* and *H*has at least two isolated vertices. Let *x* and *y* be distinct isolated vertices of *G*, *v* and *u* be distinct isolated vertices of *H*. Let $D = \{x, v\}\ and\ S = \{y, u\}$. Then by Theorem 2.10, $C = D \cup S$ is a disjoint perfect dominating set of $G + H$. Thus, $\gamma_P\gamma_P(G + H) \leq |C| = 4$.

Now, $D = \{x, v\}$ is a minimum dominating set of $G + H$ because $D\backslash\{x\}$ or $D\backslash\{v\}$ is not a dominating set of $G + H$.

Similarly, $S = \{y, u\}$ is a minimum dominating set of $G + H$ and $S \subset V(G + H)\backslash D$ is a minimum inverse dominating set of $G + H$. Thus, $C = D \cup S = \{x, v, y, u\}$ is a minimum disjoint dominating set of $G + H$, that is, $\gamma\gamma(G + H) = 4$. This implies that

$$4 = \gamma\gamma(G + H) \leq \gamma_p\gamma_p(G + H) \leq 4.$$

Accordingly, $\gamma_p\gamma_p(G + H) = 4$. ∎

The following result is a direct consequence of Corollary 2.11.

Corollary 2.12*If G and H are complete graphs, then* $\gamma_p\gamma_p(G + H) = 2$.

Definition 2.13 *Let G and H be graphs of order m and n, respectively. The corona of two graphs G and H is the graph* $G \circ H$ *obtained by taking one copy of G and m copies of H, and then joining the i-th vertex of G to every vertex of the i-th copy of H. The join of vertex v of G and a copy* H^v*of H in the corona of G and H is denoted by* $v + H^v$.

Let G be a connected graph and $x \in V(G)$. Since, $V(G)\backslash\{x\}$ is not a dominating set of $G \circ H$ for any simple graph *H*, it follows that $V(G)$ is a minimum dominating set of $G \circ H$. Thus, the following remark holds.

Remark 2.14*Let G be a connected graph and H be any graph. Then* $V(G)$ *is a minimum dominating set of* $G \circ H$.

The next result presents a characterization of disjoint perfect dominating sets in the corona of two connected graphs and gives the corresponding disjoint perfect domination number.

Theorem 2.15 *Let G and H be connected graphs. A subset* $C = D \cup S$ of $V(G \circ H)$ *is a disjoint perfect dominating set of* $G \circ H$ *with respect to D if and only if one of the following statements is satisfied.*

(i) $D = V(G)\ and\ S = \bigcup_{v \in D} S_v\ where\ S_v = \{x\}\ is\ a\ dominating\ set\ of\ H^v\ for\ each\ v \in D.$

(ii) $S = V(G)\ and\ D = \bigcup_{v \in S} D_v\ where\ D_v = \{x\}\ is\ a\ dominating\ set\ of\ H^v\ for\ each\ v \in S.$

(iii) $D = \bigcup_{v \in V(G)} D_v\ where\ D_v = \{x\}$ is a dominating set of $H^v\ for\ each\ v \in V(G)\ and\ S = \bigcup_{v \in V(G)} S_v\ where\ S_v = \{y\}\ is\ a\ dominating\ set\ of\ H^v\ for\ each\ v \in V(G)\ with\ x \neq y.$

Proof: Suppose that a proper subset $C = D \cup S$ is a disjoint perfect dominating set of $G \circ H$ with respect to D. Then D is a minimum perfect dominating set of $G \circ H$. By the definition of corona, each $u \in V(G \circ H)\backslash V(G)$ is dominated by exactly one element of $V(G)$ implies that $V(G)$ is a perfect dominating set of $G \circ H$. Since $V(G)$

is a minimum dominating set of $G \circ H$, by Remark 2.14, it follows that $V(G)$ is the minimum perfect dominating set of $G \circ H$.

Case1. If $D = V(G)$, then $S \subseteq V(G \circ H)\backslash D = \bigcup_{v \in D} V(H^v)$. Let $S = \bigcup_{v \in D} S_v$ where $S_v \subseteq V(H^v)$ *for each* $v \in D$. Since S is a dominating set of $G \circ H$, S_v must be a dominating set of H^v for each $v \in D$. Suppose that $|S_v| \geq 2$ for each $v \in D$. Let $x, y \in S_v$ for each $v \in D$. Then, $vx, vy \in E(v + H^v)$ implies that S_v is not a perfect dominating set of H^v for each $v \in D$. Thus, $S = \bigcup_{v \in D} S_v$ is not a perfect dominating set of $G \circ H$ contrary to our assumption. This implies that $S_v \ngeq 2$ and hence $|S_v| = 1$. Let $S_v = \{x\}$ for each $v \in D$. Thus, $D = V(G)$ *and* $S = \bigcup_{v \in D} S_v$ where $S_v = \{x\}$ is a dominating set of H^v for each $v \in D$. This shows statement (i).

Case2. If $D \neq V(G)$, then $S = V(G)$ and $D \subseteq V(G \circ H)\backslash S = \bigcup_{v \in S} V(H^v)$. Let $D = \bigcup_{v \in S} D_v$ where D_v is a dominating set of H^v for each $v \in S$. Since D is a minimum perfect dominating set of $G \circ H$, it follows that

$$|V(G)| = |D| \leq |S| = |V(G)|,$$

that is, $|D| = |S|$. Thus, $|D| = |\bigcup_{v \in S} D_v| = \sum_{v \in S} |D_v| = |S| \cdot |D_v| = |S|$ implies that $|D_v| = 1$. Let $D_v = \{x\}$ for each $v \in V(G)$. Thus, $S = V(G)$ and $D = \bigcup_{v \in S} D_v$ where $D_v = \{x\}$is a dominating set of H^v for each $v \in S$. This shows statement (ii).

Case3. Suppose that $D \neq V(G)$ and $S \neq V(G)$. Then, $D \subseteq \bigcup_{v \in V(G)} V(H^v)$ and $S \subseteq \bigcup_{v \in V(G)} V(H^v)$. Let $D = \bigcup_{v \in S} D_v$ where D_v is a dominating set of H^v for each $v \in S$ and let $S = \bigcup_{v \in D} S_v$ where $S_v \subseteq$ *VHv* for each *v∈D.* By using similar proofs in (i) and (ii), if follows that *D=v∈V(G)Dv* where *Dv=x* is a dominating set of H^v for each $v \in V(G)$ and $S = \bigcup_{v \in V(G)} S_v$ where $S_v = \{y\}$ is a dominating set of H^v for each $v \in V(G)$ with $x \neq y$. This shows statement (iii).

For the converse, suppose that statement (i) is satisfied. Then $D = V(G)$ and $S = \bigcup_{v \in D} S_v$ where $S_v = \{x\}$ is a dominating set of H^v for each $v \in D$. Since $D = V(G)$ is a minimum dominating set of $G \circ H$, by Remark 2.14, it follows that $S \subseteq V(G \circ H)\backslash \text{D} = \bigcup_{v \in D} S_v$ is an inverse dominating set of $G \circ H$with respect to D. Now, for each $u \in V(G \circ H)\backslash D$, there exists exactly one $v \in D$ such that $uv \in E(G \circ H)$, that is D is a minimum perfect dominating set of $G \circ H$. Further, $S_v = \{x\}$ is a dominating set of H^v implies that S_v is a perfect dominating set of H^v for each $v \in D$. Thus, $S = \bigcup_{v \in D} S_v$ is a perfect dominating set of $G \circ H$. This implies that $S \subseteq V(G \circ H)\backslash D$ is an inverse perfect dominating set of $G \circ H$ with respect to D, that is, $C = D \cup S$ is a disjoint perfect dominating set of $G \circ H$.

Suppose that statement (ii) is satisfied. Then $S = V(G)$ and $D = \bigcup_{v \in S} D_v$ where $D_v = \{x\}$ is a dominating set of H^v for each $v \in S$. Since $S = V(G)$ is a minimimum dominating set of $G \circ H$, by Remark 2.14, it follows that,

$$|D| = |\bigcup_{v \in S} D_v| = \sum_{v \in S} |D_v| = |S| \cdot |D_v| = |S| \cdot |\{x\}| = |S| \cdot 1 = |S|,$$

D is also a minimum dominating set of $G \circ H$. Thus, $S \subseteq V(G \circ H)\backslash D = \bigcup_{v \in D} S_v$ is an inverse dominating set of $G \circ H$ with respect to D. Now, $D_v = \{x\}$ is a dominating set of H^v implies that D_v is a perfect dominating set of H^v for each $v \in S$. Thus, $D = \bigcup_{v \in S} D_v$ is a perfect dominating set of $G \circ H$. Since $S = V(G)$ is a perfect dominating set of $G \circ H$, it follows that $S \subseteq V(G \circ H)\backslash D$ is an inverse perfect dominating set of $G \circ H$ with respect to D, that is, $C = D \cup S$ is a disjoint perfect dominating set of $G \circ H$.

Now, suppose that statement (iii) holds. Then $D_v = \{x\}$ is a dominating set of H^v implies that D_v is a perfect dominating set of H^v for each $v \in V(G)$. Thus, $D = \bigcup_{v \in V(G)} D_v$ is a perfect dominating set of $G \circ H$. Similarly, $S_v = \{x\}$ is a perfect dominating set of H^v for each $v \in V(G)$ implies that $S = \bigcup_{v \in V(G)} S_v$ is a perfect dominating set of $G \circ H$. Clearly, $|D| = |V(G)|$ and hence D is a minimum perfect dominating set of $G \circ H$. Since $x \neq y$, $D \cap S = \emptyset$. Thus, $S \subseteq V(G \circ H)\backslash D$ is an inverse perfect dominating set of $G \circ H$ with respect to D, that is, $C = D \cup S$ is a disjoint perfect dominating set of $G \circ H$. ■

Corollary 2.16 *Let G and H be connected graphs. Then*

$$\gamma_p\gamma_p(G \circ H) = 2 \cdot \gamma_P(G \circ H) \textit{if and only if } \gamma(H) = 1.$$

Proof: Suppose that the $\gamma_p\gamma_p(G \circ H) = 2 \cdot |V(G)|$. Let $C = D \cup S$ be a $\gamma_p\gamma_p - set$ in $G \circ H$. In view of Theorem 2.15(i), $D = V(G)$ and $S = \bigcup_{v \in D} S_v$ where $S_v = \{x\}$ is a dominating set of H^v for each $v \in D$. Clearly, $\gamma(H) = |S_v| = 1$for each $v \in D$.

For the converse, suppose that $\gamma(H) = 1$. Note that, $\gamma_p(G \circ H) = |V(G)|$. Consider the following cases.

Case1. Suppose that $D = V(G)$. Then, D is a minimum perfect dominating set of$G \circ H$. Let $S \subset V(G \circ H)\backslash D = \bigcup_{v \in D} V(H^v)$ and suppose that $S = \bigcup_{v \in D} S_v$, where $S_v \subseteq V(H^v)$ for each $v \in D$. Since $\gamma(H) =$, let $S_v = \{x\}$ be a dominating set of H^v for each $v \in D$. Then S is an inverse perfect dominating set of $G \circ H$ by Theorem 2.15(i) with respect to D. Thus,

$$\gamma_p^{-1}(G \circ H) \leq |S| = \sum_{v \in D} |S_v| = |D||S_v| = |D| = |V(G)| = \gamma_p(G \circ H).$$

Since,$\gamma_p(G \circ H) \leq \gamma_p^{-1}(G \circ H)$, it follows that $\gamma_p^{-1}(G \circ H) = \gamma_p(G \circ H)$. Thus,

$$\gamma_p\gamma_p(G \circ H) = \gamma_p(G \circ H) + \gamma_p^{-1}(G \circ H) = 2 \cdot \gamma_p(G \circ H).$$

Case2. Suppose that $S = V(G)$. Let $D = \bigcup_{v \in S} D_v$ where $D_v \subseteq V(H^v)$ for each $v \in V(G)$.Since $\gamma(H) = 1$, let $D_v = \{y\}$ be a dominating set of H^v for each $v \in S$. Then S is an inverse perfect dominating set of $G \circ H$ by Theorem 2.15(ii) with respect to D. Thus, $\gamma_p^{-1}(G \circ H) \leq |S| = |V(G)| = \gamma_p(G \circ H)$, that is, $\gamma_p^{-1}(G \circ H) = \gamma_p(G \circ H)$. Similarly, $\gamma_p\gamma_p(G \circ H) = 2 \cdot \gamma_p(G \circ H)$.

Case3. Suppose that $D \neq V(G)$ and $S \neq V(G)$. Let $D = \bigcup_{v \in V(G)} D_v$ where D_v is a dominating set of H^v for each $v \in V(G)$ and $S = \bigcup_{v \in V(G)} S_v$ where S_v is a dominating set of H^v for each $v \in V(G)$ with $x \neq y$. Since $\gamma(H) = 1$, let $S_v = \{x\}$ for each $v \in V(G)$ and let $D_v = \{y\}$ set of H^v for each $v \in S$ where $x \neq y$. Then S is an inverse perfect dominating set of $G \circ H$ by Theorem 2.15(iii). Thus, $\gamma_p^{-1}(G \circ H) \leq |S| = \sum_{v \in V(G)} |S_v| = |V(G)||S_v| = |V(G)| = \gamma_p(G \circ H)$. Therefore, $\gamma_p^{-1}(G \circ H) = \gamma_p(G \circ H)$, that is, $\gamma_p\gamma_p(G \circ H) = 2 \cdot \gamma_p(G \circ H)$. ■

III. CONCLUSION

This paper introduces a new parameter of domination in graphs called disjoint perfect domination using three different parameters of domination namely, perfect domination, inverse perfect domination, and disjoint domination in graphs. In this work, the results of disjoint perfect domination of the join and corona of two graphs were characterized by identifying the perfect dominating sets and inverse perfect dominating sets that can be found in the join and corona of two graphs. The exact disjoint perfect domination number of the join and corona of two graphs were computed and determined. Also, this study will result to new research studies such as characterizing disjoint perfect dominating sets and determining disjoint perfect domination number on graphs under binary operations like Cartesian product and lexicographic product of two graphs. This study will also prompt future exploration involving the use of disjoint perfect domination to other parameters of domination and its possible applications in the real world.

ACKNOWLEDGEMENTS

This research is funded by the Department of Science and Technology-Accelerated Science and Technology Human Resource Development Program (DOST-ASTHRDP).

REFERENCES

[1]. O. Ore. *Theory of Graphs*. American Mathematical Society, Provedence, R.I., 1962.

[2]. E.J. Cockayne, and S.T. Hedetniemi. *Towards a theory of domination in graphs*, Networks, (1977) 247-261.

[3]. R. Laskar, and S.T. Hedetniemi. *Connected domination in graphs*, Tech. Report 414, Clemson Univ., Dept. Mathematical Sci., 1983. .

[4]. E.L. Enriquez, and S.R. Canoy,Jr. *Secure Convex Domination in a Graph*. International Journal of Mathematical Analysis, Vol. 9, 2015, no. 7, 317-325.

[5]. E.J. Cockayne, B.L. Hartnell, S.T. Hedetniemi and R. Laskar, *Perfect domination in graphs*, J. Combin. Inform. System Sci. 18(1993), 136-148.

[6]. Estrada, G.M., Loquias, C.M., Enriquez, E.L., Baraca, C.S., *Perfect doubly connected domination in the join and corona of graphs*, International Journal of Latest Engineering Research and Applications, 4(7), 2019, pp 11-16.

[7]. E.L. Enriquez, V. Fernandez., T. Punzalan, J.A. Dayap, *Perfect outer-connected domination in the join andcorona of graphs*Recoletos Multidisciplinary Research Journal, 4(2), 2016.

[8]. C.S. Castañares, E.L. Enriquez. *Inverse Perfect Secure Domination in Graphs*, International Journal of Mathematics Trends and Technology, 67(8), 2022, pp 150-156.

[9]. H.R. A. Gohil, E.L Enriquez *Inverse Perfect Restrained Domination in Graphs*, International Journal of Mathematics Trends and Technology, 67(8), 2022, pp 164-170.

[10]. V.R. Kulli and S.C. Sigarkanti, *Inverse domination in graphs*, Nat. Acad. Sci. Letters, 14(1991) 473-475.

[11]. T. TamizhChelvan, T. Asir and G.S. Grace Prema, *Inverse domination in graphs*, Lambert Academic Publishing, 2013.

[12]. E.M. Kiunisala and F.P. Jamil, *Inverse domination Numbers and disjoint domination numbers of graphs under some binary operations*, Applied Mathematical Sciences, Vol. 8, 2014, no. 107, 5303-5315.

[13]. T.J. Punzalan and E.L. Enriquez, *Inverse Restrained domination in graphs*, Global Journal of Pure and Applied Mathematics, 12, No. 3(2016), pp. 2001-2009.

[14]. E.L. Enriquez, *Inverse fair domination in the join and corona of graphs*, Discrete Mathematics, Algorithms and Applications, 16(01), 2024, pp 2350003.

[15]. J.A. Ortega, E.L. Enriquez, *Super Inverse Domination in Graphs*, International Journal of Mathematics Trends and Technology, 67(7), 2021, pp 135-140.

[16]. E.M. Kiunisala, and E.L. Enriquez, *Inverse Secure Restrained Domination in the Join and Corona of Graphs*, International Journal of Applied Engineering Research, Vol. 11, 2016, no. 9, 6676-6679.

[17]. M. Hedetniemi, S.T. Hedetniemi, R.C. Laskar, L.R. Markus, P.J. Slater, *Disjoint Dominating Sets in Graphs*, Proc. of ICDM, (2006), 87-100.

[18]. R.C. Alota, and E.L. Enriquez, *On Disjoint Restrained Domination in Graphs*, Global Journal of Pure and Applied Mathematics, 12, No. 3(2016), pp. 2385-2394.

[19]. SP.G. Cajigas, E.L. Enriquez, K.E. Belleza, G.M. Estrada, C.M. Loquias. *Disjoint Restrained Domination in the Join and Corona of Graphs*, International Journal of Mathematics Trends and Technology, 67(12), 2021, pp 57-61. 13

[20]. M.D. Garol, E.L. Enriquez, K.E. Belleza, G.M. Estrada, C.M. Loquias. *Disjoint Fair Domination in the Join and Corona of Two Graphs*, International Journal of Mathematics Trends and Technology, 68(2), 2022, pp 124-132.

[21]. *Disjoint secure domination in the join of graphs*. JonecisDayap, Enrico Enriquez, Recoletos Multidisciplinary Research Journal, 4(2), 2016.

[22]. G. Chartrand and P. Zhang.*A First Course in Graph Theory*, Dover Publication, Inc., New York, 2012.

[23]. D.P. Salve and E.L. Enriquez, *Inverse perfect domination in graphs*, Global Journal of Pure and Applied Mathematics, 12, No. 1(2016) 1-10.

