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Restrained Inverse Domination in the Join and Corona of Two Graphs

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Abstract: Let G be a connected simple graph and D be a minimum dominating set of G. A dominating set $S \subseteq V(G) \setminus D$ is called an inverse dominating set of G with respect to D. An inverse dominating set S is called a restrained inverse dominating set of G if every vertex not in S is adjacent to a vertex in S and to a vertex in $V(G) \setminus S$. The restrained inverse domination number of G, denoted by $\gamma_r^{(-1)}(G)$, is the minimum cardinality of a restrained inverse dominating set of G. A restrained inverse dominating set of cardinality $\gamma_r^{(-1)}(G)$ is called $\gamma_r^{(-1)}$-set. In this paper, we initiate a study of the concept and characterize the restrained inverse dominating sets in the join and corona of two graphs.
Keywords: dominating set, inverse dominating set,join and corona,restrained dominating set, restrained inverse dominating set

I. INTRODUCTION

Let G be a connected simple graph. A set S of vertices of G is a dominating set of G if every vertex in $V(G) \setminus S$ is adjacent to some vertex in S. A minimum dominating set in a graph G is a dominating set of minimum cardinality. The cardinality of a minimum dominating set in G is called the domination number of G and is denoted by $\gamma(G)$. The concept of domination in graphs introduced by Claude Berge in 1958 and Oystein Ore in 1962 [1] is currently receiving much attention in literature. Following the article of Ernie Cockayne and Stephen Hedetniemi [2], the domination in graphs became an area of study by many researchers [3, 4, 5, 6, 7, 8].

If D is a minimum dominating set in G, then a dominating set $S \subseteq V(G) \setminus D$ is called an inverse dominating set with respect to D. The inverse domination number, denoted by $\gamma^{-1}(G)$, of G is the order of an inverse dominating set with minimum cardinality. The inverse domination in a graph was first found in the paper of Kulli [9] and studied in [10, 11, 12, 13, 14, 15, 16].

Another type of domination parameter is the restrained domination number in a graph. A restrained dominating set is defined to be a set $S \subseteq V(G)$ where every vertex in $V(G) \setminus S$ is adjacent to a vertex in S and to another vertex in $V(G) \setminus S$. The restrained domination number of G, denoted by $\gamma_r(G)$, is the smallest cardinality of a restrained dominating set of G. This was introduced by Telle and Proskurowski [17] indirectly as a vertex partitioning problem. One practical application of restrained domination is that of prisoners and guards. Here, each vertex not in the restrained dominating set corresponds to a position of a prisoner, and every vertex in the restrained dominating set corresponds to a position of a guard. To effect security, each prisoner's position is observed by a guard's position. To protect the rights of prisoners, each prisoner's position is seen by at least one other prisoner's position. To be cost effective, it is desirable to place as few guards as possible. Some studies on restrained domination in graphs can be found in [18, 19, 20, 21, 22, 23, 24].

A graph$G$ is a pair $(V(G), E(G))$, where $V(G)$is a nonempty finite set whose elements are called vertices and $E(G)$ is a set of unordered pairs of distinct elements of $V(G)$. The elements of $E(G)$ are called edges of the graph G. The number of vertices in G is called the order of G and the number of edges is called the size of G. For more graph-theoretical concepts, the readers may refer to [25].

Motivated by the concepts of restrained domination and inverse domination in graphs, the researchers introduced a new variant, that is, the restrained inverse domination in graphs. An inverse dominating set S is called a restrained inverse dominating set of G if every vertex not in S is adjacent to a vertex in S and to a vertex in $V(G) \setminus S$. The restrained inverse domination number of G, denoted by $\gamma_r^{(-1)}(G)$, is the minimum cardinality of a restrained inverse dominating set of G. A restrained inverse dominating set of cardinality $\gamma_r^{(-1)}(G)$ is called $\gamma_r^{(-1)}$-set. In this paper, the researchers have initiated a study of the concept and characterized the restrained inverse domination in the join and corona of two graphs.

II. RESULTS

Definition 2.1 *The join $G+H$ of two graphs G and H is the graph with vertex-set $V(G+H)=V(G)\cup V(H)$ and edge-set $E(G+H)=E(G)\cup E(H)\cup\{uv: u\in V(G), v\in V(H)\}$.*

Remark 2.2 *Let G and H be nontrivial connected graphs. Then $\gamma(G+H)\leq 2$.*

The following result is the characterization of a restrained inverse dominating set of $G+H$.

Theorem 2.3 *Let G and H be nontrivial connected graphs. Then a nonempty $S\subseteq V(G+H)\setminus D$ is a restrained inverse dominating set of $G+H$ with respect to a minimum dominating set D of $G+H$, if and only if S is a dominating set and one of the following is satisfied.*

(i) ($S=V(G)$ and $\gamma(H)\leq 2$) or ($S\subseteq V(G)\setminus D$ and $\gamma(G)\leq 2$).

(ii) ($S=V(H)$ and $\gamma(G)\leq 2$) or ($S\subseteq V(H)\setminus D$ and $\gamma(H)\leq 2$).

(iii) $S=S_G\cup S_H$ and $D=\{x,y\}$ where $S_G\subseteq V(G)\setminus\{x\}, \gamma(G)\neq 1$ and $S_H\subseteq V(H)\setminus\{y\}, \gamma(H)\neq 1$.

Proof: Suppose that a nonempty $S\subseteq V(G+H)\setminus D$ is a restrained inverse dominating set of $G+H$ with respect to a minimum dominating set D of $G+H$.

Case1. If $S\cap V(H)=\emptyset$, then $S\subseteq V(G)$. If $S=V(G)$, then $D\subset V(H)$ and D must be a minimum dominating set of H, by definition of D. Thus, $\gamma(H)=|D|=\gamma(G+H)\leq 2$ by Remark 2.2, that is, $\gamma(H)\leq 2$. If $S\subset V(G)$ and given that $S\subseteq V(G+H)\setminus D$, then $S\subseteq V(G)\setminus D$. This implies that $D\subset V(G)$ and hence $\gamma(G)=|D|=\gamma(G+H)\leq 2$ by Remark 2.2, that is, $\gamma(G)\leq 2$. This shows statement *(i)*.

Case2. If $S\cap V(G)=\emptyset$ then $S\subseteq V(H)$. If $S=V(H)$, then $D\subset V(G)$ and D must be a minimum dominating set of G, by definition of D. Thus, $\gamma(G)=|D|=\gamma(G+H)\leq 2$ by Remark 2.2, that is, $\gamma(G)\leq 2$. If $S\subset V(H)$ and given that $S\subseteq V(G+H)\setminus D$, then $S\subseteq V(H)\setminus D$. This implies that $D\subset V(H)$ and hence $\gamma(H)=|D|=\gamma(G+H)\leq 2$ by Remark 2.2, that is, $\gamma(H)\leq 2$. This shows statement *(ii)*.

Case3. If $S\cap V(G)\neq\emptyset$ and $S\cap V(H)\neq\emptyset$, then let $S_G=S\cap V(G)$ and $S_H=S\cap V(H)$. Then

$$\begin{aligned} S_G\cup S_H &= \big(S\cap V(G)\big)\cup\big(S\cap V(H)\big)\\ &= S\cap\big(V(G)\cup V(H)\big)\\ &= S\cap V(G+H)\\ &= S, \end{aligned}$$

that is, $S=S_G\cup S_H$. Given that D is a minimum dominating set of $G+H$, let $D=\{x,y\}$ where $x\in V(G)$ and $y\in V(H)$. Since $S\subseteq V(G+H)\setminus D$, it follows that $S_G\subseteq V(G+H)\setminus D=V(G)\setminus\{x\}$ and $S_H\subseteq V(G+H)\setminus D=V(H)\setminus\{y\}$. This shows statement *(iii)*.

For the converse, suppose that statement *(i)* is satisfied. Then ($S=V(G)$ and $\gamma(H)\leq 2$) or ($S\subseteq V(G)\setminus D$ and $\gamma(G)\leq 2$).

First, consider that $S=V(G)$ and $\gamma(H)\leq 2$. Let $\gamma(H)=|D|$. Then $D\subset V(H)$ is a minimum dominating set of H and hence of $G+H$. Thus, $S\subset V(G+H)\setminus D$ is an inverse dominating set of $G+H$ with respect to D. Since every vertex in $V(G+H)\setminus S=V(H)$ is adjacent to a vertex in $S=V(G)$ and to another vertex in $V(H)$ (because H is a nontrivial connected graph), it follows that S is a restrained dominating set of $G+H$. Hence, a nonempty $S\subseteq V(G+H)\setminus D$ is a restrained inverse dominating set of $G+H$ with respect to a minimum dominating set D of $G+H$.

Next, consider that $S\subseteq V(G)\setminus D$ and $\gamma(G)\leq 2$. Let $\gamma(G)=|D|$. Then $D\subset V(G)$ is a minimum dominating set of G and hence of $G+H$. Thus, $S\subset V(G+H)\setminus D$ is an inverse dominating set of $G+H$ with respect to D. Let $u\in V(G+H)\setminus S$. If $u\in V(G)\setminus S$, then there exists $v\in S$ such that $uv\in E(G)$ (since S is a dominating set) and $uz\in E(G+H)$ for some $z\in V(H)$. If $u\in V(H)$, then there exists $v\in S$ such that $uv\in E(G+H)$ (since S is a dominating set) and $uz\in E(H)$ for some $z\in V(H)$ (since H is a nontrivial connected graph). Thus, S is a restrained dominating set of $G+H$. Hence, a nonempty $S\subseteq V(G+H)\setminus D$ is a restrained inverse dominating set of $G+H$ with respect to a minimum dominating set D of $G+H$.

Suppose that statement *(ii)* is satisfied. Then ($S=V(H)$ and $\gamma(G)\leq 2$) or ($S\subseteq V(H)\setminus D$ and $\gamma(H)\leq 2$).

First, consider that $S=V(H)$ and $\gamma(G)\leq 2$. Let $\gamma(G)=|D|$. Then $D\subset V(G)$ is a minimum dominating set of G and hence of $G+H$. Thus, $S\subset V(G+H)\setminus D$ is an inverse dominating set $G+H$ with respect to D. Since every vertex in $V(G+H)\setminus S=V(G)$ is adjacent to a vertex in $S=V(H)$ and to another vertex in $V(G)$ (because G is a nontrivial connected graph), it follows that S is a restrained dominating set of $G+H$. Hence, a nonempty $S\subseteq V(G+H)\setminus D$ is a restrained inverse dominating set of $G+H$ with respect to a minimum dominating set D of $G+H$.

Next, consider that $S\subseteq V(H)\setminus D$ and $\gamma(H)\leq 2$. Let $\gamma(H)=|D|$. Then $D\subset V(H)$ is a minimum dominating set of H and hence of $G+H$. Thus, $S\subset V(G+H)\setminus D$ is an inverse dominating set of $G+H$ with respect to D. Let $u\in V(G+H)\setminus S$. If $u\in V(H)\setminus S$, then there exists $v\in S$ such that $uv\in E(H)$ (since S is a dominating set) and $uz\in E(G+H)$ for some $z\in V(G)$. If $u\in V(G)$, then there exists $v\in S$ such that $uv\in$

$E(G+H)$ (since S is a dominating set) and $uz \in E(G)$ for some $z \in V(G)$ (since G is a nontrivial connected graph). Thus, S is a restrained dominating set of $G+H$. Hence, a nonempty $S \subseteq V(G+H) \setminus D$ is a restrained inverse dominating set of $G+H$ with respect to a minimum dominating set D of $G+H$.

Suppose that statement *(iii)* is satisfied. Then $S = S_G \cup S_H$ and $D = \{x, y\}$ where $S_G \subseteq V(G) \setminus \{x\}, \gamma(G) \neq 1$ and $S_H \subseteq V(H) \setminus \{y\}, \gamma(H) \neq 1$. Then D is a minimum dominating set of $G+H$ (since $\gamma(G+H\neq 1$ as $\gamma G\neq 1$ and $\gamma H\neq 1)$. Thus, $S\subseteq VG+H\setminus D$ is an inverse dominating set of $G+H$. Let $u\in VG+H\setminus S$. If $u \in V(G) \setminus S_G$, then there exists $v \in S_H \subset S$ such that $uv \in E(G+H)$ and $uz \in E(G)$ for some $z \in V(G)$(since G is a nontrivial connected graph). If $u \in V(H)\setminus S_H$, then there exists $v \in S_G \subset S$ such that $uv \in E(G+H)$ and $uz \in E(H)$ for some $z \in V(H)$ (since H is a nontrivial connected graph). Thus, S is a restrained dominating set of $G+H$. Hence, a nonempty $S \subseteq V(G+H) \setminus D$ is a restrained inverse dominating set of $G+H$ with respect to a minimum dominating set D of $G+H$. ■

The following result is an immediate consequence of Theorem 2.3

Corollary 2.4*Let G and H be nontrivial connected graphs.*

$$\gamma_r^{(-1)}(G+H) = \begin{cases} 1 & if\ G = K_2 + J\ for\ any\ graph\ J, \\ & or\ H = K_2 + J\ for\ any\ graph\ J \\ 2 & if\ \gamma(G) \neq 1\ and\ \gamma(H) \neq 1. \end{cases}$$

Proof: Suppose that $G = K_2 + J$ for any graph J with $V(K_2) = \{x_1, x_2\}$. Let $S \subseteq V(G) \setminus D$ with $D = \{x_1\}$ a minimum dominating set of G. Then, by Theorem 2.3*(i)*, S is a restrained inverse dominating set of $G+H$ with respect to a minimum dominating set D of $G+H$. Thus, $\gamma_r^{(-1)}(G+H) \leq |S|$. Let $S = \{x_1\}$. Then S is a minimum dominating set of G and hence of $G+H$. Thus, $|S| = \gamma(G+H) \leq \gamma_r^{(-1)}(G+H) \leq |S|$, that is, $\gamma_r^{(-1)}(G+H) = |S| = 1$. Similarly, if $H = K_2 + J$ for any graph J, then $\gamma_r^{(-1)}(G+H) = 1$.

Next, suppose that $\gamma(G) \neq 1$ and $\gamma(H) \neq 1$. Let $D = \{x, y\}$ with $x \in V(G)$ and $y \in V(H)$. Then $V(G) \setminus \{x\} \neq \emptyset$ and $V(H) \setminus \{y\} \neq \emptyset$ since G and H are nontrivial connected graphs. Let $S_G \subset V(G) \setminus \{x\}$ and $S_H \subset V(H) \setminus \{y\}$, that is, $S = S_G \cup S_H \subset V(G+H) \setminus D$. By Theorem 2.3*(iii)*, S is a restrained inverse dominating set of $G+H$ with respect to a minimum dominating set D of $G+H$. Thus, $\gamma_r^{(-1)}(G+H) \leq |S|$. Clearly, $S = \{v, v'\}$ (with $S_G = \{v\}, v \neq x$ and $S_H = \{v'\}, v' \neq y$) is a dominating set of $G+H$ and a minimum dominating set of $G+H$ (since $\gamma(G) \neq 1$ and $\gamma(H) \neq 1$). Thus, $|S| = \gamma(G+H) \leq \gamma_r^{(-1)}(G+H) \leq |S|$, that is, $\gamma_r^{(-1)}(G+H) = |S| = 2$. ■

Definition 2.5*The corona of two graphs G and H, denoted by $G \circ H$ is the graph obtained by taking one copy of G of order n and n copies of H, and then joining the $i-th$ copy of H. For every $v \in V(G)$, we denote by H^v the copy of H whose vertices are joined or attached to the vertex v.*

Remark 2.6*Let G and H be nontrivial connected graphs. Then $V(G)$ is a minimum dominating set of $G \circ H$.*

The following results are needed for the characterization of the restrained inverse dominating sets in the corona of two graphs.

Lemma 2.7*Let G and H be nontrivial connected graphs. If $D = V(G)$ and $S = \bigcup_{v\in V(G)} S_v$ where S_v is a dominating set of H^v for each $v \in V(G)$, then a nonempty $S \subseteq V(G \circ H) \setminus D$ is a restrained inverse dominating set of $G \circ H$ with respect to a minimum dominating set D.*

Proof: Let G and H be nontrivial connected graphs. Then $D = V(G)$ is a minimum dominating set of $G \circ H$ by Remark 2.6. Further, $S = \bigcup_{v\in V(G)} S_v$ where S_v is a dominating set of H^v for each $v \in V(G)$, is an inverse dominating set of $G \circ H$ with respect to D. Let $y \in V(G \circ H) \setminus S$. If $y \in V(G)$, then there exists $x \in S_y$ such that $xy \in E(y + H^v) \subset E(G \circ H)$ and $yz \in E(G \circ H)$ for some $z \in V(G)$ (since G is a nontrivial connected graph). If $y \in V(H^v) \setminus S_v$ for each $v \in V(G)$, then there exists $x \in S_v$ such that $xy \in E(H^v) \subset E(G \circ H)$ for each $v \in V(G)$ (since S_v is a dominating set of H^v for each $v \in V(G)$). In any case, S is a restrained dominating set of $G \circ H$. Accordingly, a nonempty $S \subseteq V(G \circ H) \setminus D$ is a restrained inverse dominating set of $G \circ H$ with respect to a minimum dominating set D. ■

Lemma 2.8*Let G and H be nontrivial connected graphs. If $D = \bigcup_{v\in V(G)} D_v$ where $D_v = \{x\}$ is a dominating set of H^v for each $v \in V(G)$, and $S = \bigcup_{v\in V(G)} S_v$where $S_v \subseteq V(H^v) \setminus D_v$ is a dominating set of $H^v$for each $v \in V(G)$, then a nonempty $S \subseteq V(G \circ H) \setminus D$ is a restrained inverse dominating set of $G \circ H$ with respect to a minimum dominating set D.*

Proof: Suppose that $D = \bigcup_{v\in V(G)} D_v$ where $D_v = \{x\}$ is a dominating set of H^v for each $v \in V(G)$. Then

$$\begin{aligned}|D| &= \left|\bigcup_{v\in V(G)} D_v\right| \\ &= \sum_{v\in V(G)} |D_v| \\ &= |V(G)|\cdot|D_v| \\ &= |V(G)|\cdot|\{x\}| \\ &= |V(G)|\cdot 1 = |V(G)|.\end{aligned}$$

This implies that D is a minimum dominating set of $G \circ H$ by Remark 2.6. Further, $S = \bigcup_{v\in V(G)} S_v$ where $S_v \subseteq V(H^v) \setminus D_v$ is a dominating set of H^v for each $v \in V(G)$, implies that S is an inverse dominating set of $G \circ H$ with respect to D. Let $y \in V(G \circ H) \setminus S$. If $y = x \in D_v$, then there exists $z \in S_v$ for each $v \in V(G)$ such that $xz \in E(H^v) \subset E(G \circ H)$ (since D_v is a dominating set of H^v for each $v \in V(G)$) and $yv \in E(v + H^v) \subset E(G \circ H)$. If $y \in V(G)$, then there exists $z \in S_y$ such that $yz \in E(y + H^y) \subset E(G \circ H)$ and $xy \in E(y + H^y) \subset V(G \circ H)$ where $x \in D_y$ for each $y \in V(G)$. In any case, S is a restrained dominating set of $G \circ H$. Accordingly, a nonempty $S \subseteq V(G \circ H)\setminus D$ is a restrained inverse dominating set of $G \circ H$ with respect to a minimum dominating set D. ■

Lemma 2.9*Let G and H be nontrivial connected graphs. If $D = \bigcup_{v\in V(G)} D_v$ where $D_v = \{x\}$ is a dominating set of H^v for each $v \in V(G)$, and $S = V(G) \cup X$ where $X \subseteq \bigcup_{v\in V(G)} (V(H^v) \setminus (D_v \cup B)$ for some $B \subset V(H^v) \setminus D_v$, then a nonempty $S \subseteq V(G \circ H) \setminus D$ is a restrained inverse dominating set of $G \circ H$ with respect to a minimum dominating set D.*

Proof: Suppose that $D = \bigcup_{v\in V(G)} D_v$ where $D_v = \{x\}$ is a dominating set of H^v for each $v \in V(G)$, then D is a minimum dominating set of $G \circ H$ by the proof of Lemma 2.8. Further, $S = V(G) \cup X$ where $X \subset \bigcup_{v\in V(G)} (V(H^v) \setminus (D_v \cup B)$ for some $B \subset V(H^v) \setminus D_v$, implies that S is an inverse dominating set of $G \circ H$ with respect to D. Let $y \in V(G \circ H) \setminus S$. If $y = x \in D_v$, then there exists $z \in X$ such that $xz \in E(H^v) \subset E(G \circ H)$ (since D_v is a dominating set of H^v for each $v \in V(G)$) and $yu \in E(H^v) \subset E(G \circ H)$ for some $u \in B$. If $y \in B$, then there exists $z \in X$ such that $yz \in E(H^v) \subset E(G \circ H)$ for each $v \in V(G)$ and $yx \in E(H^v) \subset V(G \circ H)$ where $x \in D_v$ for each $v \in V(G)$. In any case, S is a restrained dominating set of $G \circ H$. Accordingly, a nonempty $S \subseteq V(G \circ H) \setminus D$ is a restrained inverse dominating set of $G \circ H$ with respect to a minimum dominating set D. ■

The following result is the characterization of a restrained inverse dominating set of $G \circ H$.

Theorem 2.10*Let G and H be nontrivial connected graphs. Then a nonempty $S \subseteq V(G \circ H) \setminus D$ is a restrained inverse dominating set of $G \circ H$ with respect to a minimum dominating set D if and only if one of the following is satisfied.*

(i) $D = V(G)$ and $S = \bigcup_{v\in V(G)} S_v$ where S_v is a dominating set of H^v for each $v \in V(G)$.

(ii) $D = \bigcup_{v\in V(G)} D_v$ where $D_v = \{x\}$ is a dominating set of H^v for each $v \in V(G)$, and

a) $S = \bigcup_{v\in V(G)} S_v$ where $S_v \subseteq V(H^v) \setminus D_v$ is a dominating set of H^v for each $v \in V(G)$, or

b) $S = V(G) \cup X$ where $X \subseteq \bigcup_{v\in V(G)} (V(H^v) \setminus D_v)$.

Proof: Suppose that a nonempty $S \subseteq V(G \circ H) \setminus D$ is a restrained inverse dominating set of $G \circ H$ with respect to a minimum dominating set D.

Case1. If $D = V(G)$, then $S \subseteq V(G \circ H) \setminus D = \bigcup_{v\in V(G)} V(H^v)$. Let $S = \bigcup_{v\in V(G)} S_v$ where $S_v \subseteq V(H^v)$ and S_v is a dominating set of H^v for each $v \in V(G)$. This shows statement *(i).*

Case2. If $D \neq V(G)$, then $D \subset V(G \circ H) \setminus V(G) = \bigcup_{v\in V(G)} V(H^v)$. Let $D = \bigcup_{v\in V(G)} D_v$ where $D_v = \{x\} \subset V(H^v)$ and D_v is a dominating set of H^v for each $v \in V(G)$.

Subcase1. If $S \subseteq \bigcup_{v\in V(G)} V(H^v) \setminus D_v$, then let $S = \bigcup_{v\in V(G)} S_v$ where $S_v \subseteq V(H^v) \setminus D_v$ is a dominating set of H^v for each $v \in V(G)$. This shows statement *(ii)a)*

Subcase2. If $S \not\subseteq \bigcup_{v\in V(G)} V(H^v) \setminus D_v$, then

$$S \subseteq V(G \circ H) \setminus D = V(G \circ H) \setminus \left(\bigcup_{v \in V(G)} D_v \right)$$
$$= V(G) \bigcup \left(\bigcup_{v \in V(G)} (V(H^v) \setminus D_v) \right)$$

Let $X \subseteq \bigcup_{v \in V(G)} (V(H^v) \setminus D_v)$ such that $S = V(G) \cup X$. This shows statement *(ii)b).*

For the converse, suppose that statement *(i)* is satisfied. Then $D = V(G)$ and $S = \bigcup_{v \in V(G)} S_v$ where S_v is a dominating set of H^v for each $v \in V(G)$. By Lemma 2.7, a nonempty $S \subseteq V(G \circ H) \setminus D$ is a restrained inverse dominating set of $G \circ H$ with respect to a minimum dominating set D.

Suppose that statement *(ii)a)* is satisfied. Then $D = \bigcup_{v \in V(G)} D_v$ where $D_v = \{x\}$ is a dominating set of H^v for each $v \in V(G)$, and $S = \bigcup_{v \in V(G)} S_v$ where $S_v \subseteq V(H^v) \setminus D_v$ is a dominating set of H^v for each $v \in V(G)$. By Lemma 2.8, a nonempty $S \subseteq V(G \circ H) \setminus D$ is a restrained inverse dominating set of $G \circ H$ with respect to a minimum dominating set D.

Suppose that statement *(ii)b)* is satisfied. Then $D = \bigcup_{v \in V(G)} D_v$ where $D_v = \{x\}$ is a dominating set of H^v for each $v \in V(G)$, and $S = V(G) \cup X$ where $X \subseteq \bigcup_{v \in V(G)} (V(H^v) \setminus D_v)$. By Lemma 2.9, a nonempty $S \subseteq V(G \circ H) \setminus D$ is a restrained inverse dominating set of $G \circ H$ with respect to a minimum dominating set D. ■

Corollary 2.11*Let G and H be nontrivial connected graphs. Then*

$$\gamma_r^{(-1)}(G \circ H) = |V(G)| \cdot \gamma(H).$$

Proof: In view of Remark 2.6, $V(G)$ is a minimum dominating set of $G \circ H$. Let $D = V(G)$ and $S \subseteq V(G \circ H) \setminus D = \bigcup_{v \in V(G)} (V(H^v))$ is an inverse dominating set of $G \circ H$ with respect to D. Let $S = \bigcup_{v \in V(G)} S_v$ where $S_v \subseteq V(H^v)$ and S_v is a dominating set of H^v for each $v \in V(G)$. By Theorem 2.10, S is a restrained inverse dominating set of $G \circ H$. Thus,

$$\gamma_r^{(-1)}(G \circ H) \leq |S|$$
$$= \left| \bigcup_{v \in V(G)} S_v \right|$$
$$= \sum_{v \in V(G)} |S_v|$$
$$= |V(G)| \cdot |S_v|, \text{ that is,}$$

$\gamma_r^{(-1)}(G \circ H) \leq |V(G)| \cdot |S_v|$ for all dominating set $S_v \subseteq V(H^v)$ for each $v \in V(G)$. Since there exists a dominating set $S_v^{'} \subseteq V(H^v)$ such that $\gamma(H^v) = |S_v^{'}|$ for all $v \in V(G)$, it follows that $\gamma_r^{(-1)}(G \circ H) = |V(G)| \cdot |S_v^{'}| = |V(G)| \cdot \gamma(H)$. Thus, $\gamma_r^{(-1)}(G \circ H) = |V(G)| \cdot \gamma(H)$. ■

III. CONCLUSION

As a result of investigating the combination of inverse domination and restrained domination in graphs, a new domination parameter called the restrained inverse domination number of a graph G was introduced. In this paper, the restrained inverse dominating set of the join and corona of two graphs were characterized and the restrained inverse domination number for these graphs were determined.This study contributes valuable knowledge to the existing theories of domination in graphs being applied to real world situations such as in solving bus routing problems, communication problems, data structures, electrical networking, defense and locations strategies and others. Further investigations on the properties and bounds of this parameter are necessary in obtaining more substantial results for the development of this study and of the theory of domination in general. In view of this,the researchers would like to recommendexploring the characterization of the restrained inverse dominating setsof graphs under the lexicographic product, and Cartesian product as this is a promising extension to this study.

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