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Disjoint Perfect Secure Domination in the Join and Corona of Graphs

Renelyn B. Udtohan[1], Enrico L. Enriquez[2]
[1,2]Department of Computer, Information Science and Mathematics
School of Arts and Sciences
University of San Carlos, 6000 Cebu City, Philippines

Abstract: Let G be a graph. A dominating set $D \subseteq V(G)$is called a secure dominating set of G if for each vertex $u \in V(G) \setminus D$, there exists a vertex $v \in D$ such that $uv \in E(G)$ and the set $(D \setminus \{v\}) \cup \{u\}$ is a dominating set of G. If every $u \in V(G) \setminus D$ is adjacent to exactly one vertex in D, then D is a perfect secure dominating set of G. Let D be a minimum perfect secure dominating set of G. If $S \subseteq V(G) \setminus D$ is a perfect secure dominating set of G, then S is called an inverse perfect secure dominating set of G with respect to D. A disjoint perfect secure dominating set of G is the set $C = D \cup S \subseteq V(G)$. Furthermore, the disjoint perfect secure domination number, denoted by $\gamma_{ps}\gamma_{ps}(G)$, is the minimum cardinality of a disjoint perfect secure dominating set of G. A disjoint perfect secure dominating set of cardinality $\gamma_{ps}\gamma_{ps}(G)$ is called $\gamma_{ps}\gamma_{ps}$*-set*. In this paper, we initiate a study of the concept of disjoint perfect secure domination in graphs and characterize this type of domination in graphs under some binary operations, namely the join and corona of two graphs.

Keywords: corona of two graphs, disjoint dominating set, dominating set, inverse dominating set, inverse perfect secure dominating set, join of two graphs, perfect secure dominating set

I. INTRODUCTION

Domination is one of the branches of graph theory that gained significant attention over the years. The concept of domination in graphs was first introduced by Berge [1], while the terminologies of dominating set and domination number were formally defined by Ore [2]. These pioneering concepts have motivated researchers to study other types of domination parameters. One of these types is the secure domination in graphs. The concept of secure domination was formally introduced by Cockayne, Favaron, and Mynhardt [3]. A dominating set $D \subseteq V(G)$ is called a *secure* dominating set of G if for each vertex $u \in V(G) \setminus D$, there exists a vertex $v \in D$such that $uv \in E(G)$ and the set $(D \setminus \{v\}) \cup \{u\}$ is a dominating set of G. More topics on secure dominating sets can be read from [4, 5, 6, 7, 8].

Another variant of domination in graphs is the perfect dominating set. A dominating set $S \subseteq V(G)$ is called a *perfect dominating set* of G if each $u \in V(G) \setminus S$ is dominated by exactly one element of S. This type of domination was introduced by Cockayne et.al [9]. Additional topics on perfect dominating sets can be read from [10, 11].

A different variant of domination in graphs is the inverse dominating set. The dominating set $S \subseteq V(G) \setminus D$ is called an *inverse dominating set* of G with respect to a minimum dominating set D. The inverse domination in graphs was first introduced in the paper of Kulli [12] and can be further read in [13, 14, 26, 15, 16, 17].

A further variant of domination is the disjoint domination in graphs. In [18], Hedetniemi et al. defined the *disjoint domination* as $(G) = min\{|S1| + |S2| : S1$ and $S2$ are disjoint dominating sets of $G\}$. The two disjoint dominating sets whose union has cardinality $\gamma\gamma(G)$ is a $\gamma\gamma$*-pair* of G. For further exploration of the disjoint domination in graphs, one can refer to [19, 20, 21, 22]. Similarly, a comprehensive understanding of general concepts in graph domination can be obtained from [23, 24].

A graph $G = (V(G), E(G))$ is a pair where $V(G)$ denotes a finite nonempty set called the *vertex set* of G and $E(G)$ is a set of unordered pairs $\{u, v\}$ (or simply uv) of distinct elements from $V(G)$ called the *edge set* of G. The elements of $V(G)$ are called *vertices* and the cardinality, denoted as $|V(G)|$, of $V(G)$ is the *order* of $V(G)$. The elements of $E(G)$ are called *edges* and the cardinality, denoted as $|E(G)|$, of $E(G)$ is the *size* of G. If and the cardinality, denoted as $|V(G)|$, of G is the *order* of $V(G) = 1$ then G is called a *trivial* graph. If $E(G) = \emptyset$, then G is called an *empty* graph. The *open neighborhood* of a vertex x, denoted by $N_G(x)$, is the set of all vertices adjacent to x in G. The elements of $N_G(x)$ are called *neighbors* of x. Similarly, $N_G(S)$, denotes the neighborhood of the set S and is the collection of all vertices adjacent to some vertex in S. The *closed neighborhood* of a vertex v is the set $N[v] = N(v) \cup \{v\}$.

The perfect secure domination in graphs was introduced by Rashmi, Arumugam, Bhutani, and Gartland [25] where they explored the combination of two domination parameters, namely perfect domination and secure domination in graphs. The authors considered a graph $G = (V(G), E(G))$ where a subset D of vertices in $V(G)$ is called a *perfect secure dominating set* of G if, for every vertex $v \in V(G) \setminus D$, there exists a unique vertex $u \in D$ such that u and v are adjacent. In addition, the set obtained by $(D \setminus \{v\}) \cup \{u\}$ must be dominating set of G. The minimum number of vertices required to form a perfect secure dominating set of G is defined as the perfect secure domination number of G and is denoted by $\gamma_{ps}(G)$.

Castañares and Enriquez [26] initiated the study on the inverse perfect secure domination in graphs. Let D be a minimum perfect secure dominating set of G. If $S \subseteq V(G) \setminus D$ is a perfect secure dominating set of G, then S is called an *inverse perfect secure* dominating set of G with respect to D. The inverse perfect secure domination number of G, denoted by $\gamma_{ps}^{-1}(G)$ is the minimum cardinality of an inverse perfect secure dominating set of G.

The concepts of inverse perfect secure domination and disjoint domination in graphs have motivated the researchers to initiate a study on another domination variant called the disjoint perfect secure domination in graphs. Let D be a minimum perfect secure dominating set of G. If $S \subseteq V(G) \setminus D$ is a perfect secure dominating set of G, then S is called an inverse perfect secure dominating set of G with respect to D. A *disjoint perfect secure* dominating set of G is the set $C = D \cup S \subseteq V(G)$. Furthermore, the disjoint perfect secure domination number, denoted by $\gamma_{ps}\gamma_{ps}(G)$, is the minimum cardinality of a disjoint perfect secure dominating set of G. A disjoint perfect secure dominating set with cardinality $\gamma_{ps}\gamma_{ps}(G)$is called $\gamma_{ps}\gamma_{ps}$-set. In this paper, we initiate a study of the concept of disjoint perfect secure domination in graphs and give some important results.

II. RESULTS

Since the $\gamma_p^{-1}(G)$ does not always exist in a connected nontrivial graph G by Salve et.al. [16], the researchers introduce $\mathcal{DPS}(G)$ as a family of all graphs with inverse perfect secure dominating set and disjoint perfect secure dominating set. Thus, for the purpose of this study, it is assumed that all connected nontrivial graphs considered belong to the family $\mathcal{DPS}(G)$.

Theorem 2.1. [16] Let G be a connected graph of order $n \geq 2$. Then $\gamma_p^{-1}(G) = 1$ if and only if $G = K_1 + H$ where $\gamma(H) = 1$.

Remark 2.2. Let G and H be any graphs. Then $\gamma_{ps}\gamma_{ps}(G + H) \neq 1$.

Remark 2.3. Let G be a nontrivial connected simple graph.

$$\gamma_{ps}\gamma_{ps}(P_n) = \begin{cases} n, & if\ n\ is\ even \\ none, & ifotherwise. \end{cases}$$

Remark 2.4. Let G be a nontrivial connected simple graph.

$$\gamma_{ps}\gamma_{ps}(C_n) = \begin{cases} 2, & if\ n = 3 \\ n, & if\ n = 4k, k \in \mathbb{Z}^+. \end{cases}$$

Corollary 2.5. The difference between$\gamma_{ps}\gamma_{ps}(G) - \gamma(G)$ can be made arbitrarily large.

Proof: Let $G = P_n$ where $n = 2k$ for all positive integer k. Then

$$\gamma_{ps}\gamma_{ps}(G) - \gamma(G) = n - \frac{n}{2} = \frac{n}{2},$$

can be made arbitrarily large as n increases.■

Remark 2.6. $\gamma_{ps}(K_n) = 1$ for all positive integer $n \geq 2$.

Definition 2.7. The join of two graphs G and H is the graph $G + H$ with vertex-set $V(G + H) = V(G) \cup V(H)$ and edge-set

$$E(G + H) = E(G) \cup E(H) \cup \{uv: u \in V(G), v \in V(H)\}.$$

Each of the following Lemma is needed for the characterization of a disjoint perfect secure dominating set of $G + H$.

Lemma 2.8. Let G and H be nontrivial graphs, $D \subset V(G + H)$ and $S \subset V(G + H) \setminus D$. If $D = \{x\}$ and $S = \{y\}$ are secure dominating sets of G and H is a complete graph, then a nonempty set $C = D \cup S$ is a disjoint perfect secure dominating set of $G + H$.

Proof: Suppose that $D = \{x\}$ and $S = \{y\}$ are secure dominating sets of G and H is a complete graph. Let $u \in V(G + H) \setminus D$. Then $xu \in E(G)$ and $(D \setminus \{x\}) \cup \{u\} = \{u\}$ is a dominating set of G since D is a secure dominating set of G. If $u \in V(H)$, then $\{u\}$ dominating set of $G + H$ since H is a complete graph. Thus, D is a secure dominating set of $G + H$. Similarly, S is a secure dominating set of $G + H$. Clearly, $D = \{x\}$ and $S = \{y\}$ are perfect dominating sets of $G + H$. Hence, D and S are perfect secure dominating sets of $G + H$. Since $S \subset V(G + H) \setminus D$, it follows that S is an inverse perfect secure dominating set of $G + H$ with respect to D. Thus, $C = D \cup S$ is a disjoint perfect secure dominating set of $G + H$.■

Lemma 2.9. Let G and H be nontrivial graphs, $D \subset V(G + H)$ and $S \subset V(G + H) \setminus D$. If $D = \{x\}$ and $S = \{y\}$ are secure dominating sets of H and G is a complete graph, then a nonempty set $C = D \cup S$ is a disjoint perfect secure dominating set of $G + H$.

Proof: Suppose that $D = \{x\}$ and $S = \{y\}$ are secure dominating sets of H and G is a complete graph. Let $u \in V(G + H) \setminus D$. Then $xu \in E(H)$ and $(D \setminus \{x\}) \cup \{u\} = \{u\}$ is a dominating set of H since D is a secure dominating set of H. If $u \in V(G)$, then $\{u\}$ dominating set of $G + H$ since G is a complete graph. Thus, D is a secure dominating set of $G + H$. Similarly, S is a secure dominating set of $G + H$. Clearly, $D = \{x\}$ and $S = \{y\}$ are perfect dominating sets of $G + H$. Hence, D and S are perfect secure dominating sets of $G + H$. Since $S \subset V(G + H) \setminus D$, it follows that S is an inverse perfect secure dominating set of $G + H$ with respect to D. Thus, $C = D \cup S$ is a disjoint perfect secure dominating set of $G + H$.■

Lemma 2.10. Let G and H be nontrivial graphs, $D \subset V(G + H)$ and $S \subset V(G + H) \setminus D$. If $D = \{x\}$ and $S = \{y\}$ are subsets of complete graphs H and G respectively, then a nonempty set $C = D \cup S$ is a disjoint perfect secure dominating set of $G + H$.

Proof: Suppose that $D = \{x\}$ and $S = \{y\}$ are subsets of complete graphs H and G respectively. Then D and S are perfect secure dominating sets of $G + H$ by Remark 2.6. Further, G and H are perfect secure dominating sets of $G + H$ because G and H are complete graphs. Since $S \subset V(G + H) \setminus D$, it follows that S is an inverse perfect secure dominating set of $G + H$ with respect to D. Thus, $C = D \cup S$ is a disjoint perfect secure dominating set of $G + H$.■

The following result is the characterization of a disjoint perfect secure dominating set of $G + H$.

Theorem 2.11. Let G and H be simple nontrivial graphs, $D \subset V(G + H)$ and $S \subset V(G + H) \setminus D$. Then a nonempty set $C = D \cup S$ is a disjoint perfect secure dominating set of $G + H$ if and only if one of the following is satisfied:

(i) $D = \{x\}$ and $S = \{y\}$ are secure dominating sets of G, and H is a complete graph.
(ii) $D = \{x\}$ and $S = \{y\}$ are secure dominating sets of H, and G is a complete graph.
(iii) $D = \{x\}$ and $S = \{y\}$are subsets of complete graphs G and H, respectively.

Proof: Suppose that a nonempty set $C = D \cup S$ is a disjoint perfect secure dominating set of $G + H$ where $D \subset V(G + H)$ and $S \subset V(G + H) \setminus D$. Then D is a minimum perfect secure dominating set of $G + H$ and S is an inverse perfect secure dominating set with respect to D. Consider the following cases.

Case 1. Suppose that $D \cap V(H) = \emptyset$ and $S \cap V(H) = \emptyset$. Then $D, S \subset V(G)$, that is, D and S are dominating sets of G and perfect secure dominating sets of $G + H$. If $|D| \neq 1$, then $|D| \geq 2$, that is, for every $u \in V(H)$, $uv \in E(G + H)$ for all $v \in S$. This contradicts the definition of a perfect dominating set. Hence, $|D| = 1$ and let $D = \{x\}$. If D is not a secure dominating set of G, then D being not a secure dominating set of $G + H$ is immediate. Thus, D must be a secure dominating set of G. Similarly, $S = \{y\}$ must be a secure dominating set of

G. Hence, $D = \{x\}$ and $S = \{y\}$ are secure dominating sets of G. If H is not a complete graph, then there exists $u \in V(H)$ such that $uv \notin E(H)$ for all $v \in V(H) \setminus \{u\}$. Further, $(D \setminus \{x\}) \cup \{u\} = \{u\}$ is not a dominating set of $G + H$. This is contrary to the assumption that D is a secure dominating set of $G + H$. Hence, $\{u\}$ must be a dominating set of H for all $u \in V(H)$, that is, H is a complete graph. This shows statement (i).

Case 2. Suppose that $D \cap V(H) = \emptyset$ and $S \cap V(H) = \emptyset$. Then $D, S \subset V(H)$, that is, D and S are dominating sets of H and perfect secure dominating sets of $G + H$. If $|D| \neq 1$, then $|D| \geq 2$, that is, for every $u \in V(G)$, $uv \in E(G + H)$ for all $v \in S$. This contradicts the definition of a perfect dominating set. Hence, $|D| = 1$ and let $D = \{x\}$. If D is not a secure dominating set of H, then D being not a secure dominating set of $G + H$ is immediate. Thus, D must be a secure dominating set of H. Similarly, $S = \{y\}$ must be a secure dominating set of H. Hence, $D = \{x\}$ and $S = \{y\}$ are secure dominating sets of H. If G is not a complete graph, then there exists $u \in V(G)$ such that $uv \notin E(G)$ for all $v \in V(G) \setminus \{u\}$. Further, $(D \setminus \{x\}) \cup \{u\} = \{u\}$ is not a dominating set of $G + H$. This is contrary to the assumption that D is a secure dominating set of $G + H$. Hence, $\{u\}$ must be a dominating set of G for all $u \in V(G)$, that is, G is a complete graph. This shows statement (ii).

Case 3. Suppose that $D \cap V(H) = \emptyset$ and $S \cap V(H) = \emptyset$. Then $D, S \subset V(G)$, that is, D and S are dominating sets of G and H respectively. If $|D| \neq 1$, then D is not a perfect dominating set of $G + H$ contrary to the definition of D. Similarly, if $|S| \neq 1$, then S not a perfect dominating set of $G + H$ contrary to the definition of S. Thus, $|D| = 1$ and $|S| = 1$. Let $D = \{x\}$ and $S = \{y\}$. Since by definition, D and S are secure dominating sets of G and H respectively, it follows that G and H are complete graphs by similar arguments in the proofs of statements (i) and (ii).Therefore, $D = \{x\}$ and $S = \{y\}$ are subsets of complete graphs G and H respectively, showing statement (iii).

For the converse, suppose that statement (i) is satisfied, that is, $D = \{x\}$ and $S = \{y\}$ are secure dominating sets of G, and H is a complete graph. By Lemma 2.8, $C = D \cup S$ is a disjoint perfect secure dominating set of $G + H$.

Suppose that statement (ii) is satisfied, that is, $D = \{x\}$ and $S = \{y\}$ are secure dominating sets of H, and G is a complete graph. By Lemma 2.9, $C = D \cup S$ is a disjoint perfect secure dominating set of $G + H$.

Suppose that statement (iii) is satisfied, that is, $D = \{x\}$ and $S = \{y\}$ are subsets of complete graphs $G$and H respectively. By Lemma 2.10, $C = D \cup S$ is a disjoint perfect secure dominating set of $G + H$. This complete the proofs.■

The next result is an immediate consequence of Theorem 2.11.

Corollary 2.12. Let $G$and H be nontrivial graphs. Then $\gamma_{ps}\gamma_{ps}(G + H) = 2$ if and only if G and H are complete graphs.

Proof : Suppose that $\gamma_{ps}\gamma_{ps}(G + H) = 2$. Let $C = D \cup S = \{x, y\}$. Then D and S are perfect secure dominating setsof $G + H$.

Case 1. Consider $D = \{x\}$ and $S = \{y\}$ are subsets of G. Then G is a complete graph by Remark 2.6. Let $u \in V(G + H) \setminus D$. Then $(D \setminus \{x\}) \cup \{u\} = \{u\}$ is a dominating set of $G + H$, that is, $\{u\}$ is a dominating set of for all $u \in V(H)$. Hence, H is a complete graph. Similarly, if $V(G + H) \setminus S$, then H is a complete graph.

Case 2. Consider $D = \{x\}$ and $S = \{y\}$ are subsets of H. Then H is a complete graph by Remark 2.6. Let $u \in V(G + H) \setminus D$. Then $(D \setminus \{x\}) \cup \{u\} = \{u\}$ is a dominating set of $G + H$, that is, $\{u\}$ is a dominating set of for all $u \in V(H)$. Hence, H is a complete graph. Similarly, if $V(G + H) \setminus S$, then H is a complete graph.

Case 3. Consider $D = \{x\}$ and $S = \{y\}$ are subsets of G and H respectively. Then G and H are complete graphs by Remark 2.6.

For the converse, suppose that G and H are complete graphs. Then $\gamma_{ps}(G) = 1$ and $\gamma_{ps}(H) = 1$ by Remark 2.6. $D = \{x\}$ and $S = \{y\}$ are subsets of G and H respectively. By Theorem 2.11 (iii), $C = D \cup S = \{x, y\}$ is a disjoint perfect secure dominating set of $G + H$. Thus, $\gamma_{ps}\gamma_{ps}(G + H) \leq |C| = 2$, that is, $\gamma_{ps}\gamma_{ps}(G + H)$

is either 1 or 2. Since $\gamma_{ps}\gamma_{ps}(G+H) \neq 1$, by Remark 2.2, it follows that $\gamma_{ps}\gamma_{ps}(G+H) = 2$. This complete the proofs.■

Definition 2.13. Let G and H be graphs of order m and n, respectively. The corona of G and H, denoted by $G \circ H$, is the graph obtained by taking one copy of G and m copies of H, and then joining the ith vertex of G to every vertex of the ith copy of H. The join of vertex v of G and a copy of H^v of H in the corona of G and H is denoted by $v + H^v$.

Let G be a connected graph and $x \in V(G)$. Since $V(G) \setminus \{x\}$is not a dominating set of $G \circ H$ for any simple graph H, it follows that $V(G)$is a minimum dominating set of $G \circ H$. Thus, the following remark holds.

Remark 2.14. Let G be a connected graph and H be any graph. Then $V(G)$ is a minimum perfect dominating set of $G \circ H$.

Remark 2.15. Let G be a connected graph and H be a complete graph. Then $\gamma_{ps}\gamma_{ps}(G \circ H) \neq |V(G)|$.

Lemma 2.16. Let G and H be connected nontrivial graphs. If $D = V(G)$ and $S = \bigcup_{v\in D} S_v$ where $S_v = \{x\}$ is a secure dominating set of H^v for each $v \in D$, then a subset $C = D \cup S$ of $V(G \circ H)$ is a disjoint perfect secure dominating set of $G \circ H$ with respect to D.

Proof: $D = V(G)$ is a minimum perfect dominating set of $G \circ H$ by Remark 2.14. Since every $u \in V(G \circ H) \setminus D = \bigcup_{v\in D} V(H^v)$ is dominated by exactly one $v \in D$, it follows that D is a minimum perfect dominating set of $G \circ H$. Now, $S \subseteq V(G \circ H) \setminus D = \bigcup_{v\in D} V(H^v)$ is an inverse dominating set of $G \circ H$ with respect to D. Let $S_v \subset V(H^v)$ such that $S = \bigcup_{v\in D} S_v$ where $S_v = \{x\}$ is a secure dominating set of H^v for each $v \in D$. Each $u \in V(H^v)$ is dominated by only $x \in S_v$ for all $v \in D$. Thus, S is a perfect secure dominating set of G ∘ H. By Remark 2.6, for each $v \in D$, H^v is a complete subgraph of $G \circ H$, that is, H is a complete graph. This implies that for every $u \in V(G \circ H) \setminus D, uv \in E(G \circ H)$ and $(D \setminus \{v\}) \cup \{u\}$ is a dominating set of $G \circ H$. Thus, D is a secure dominating set of $G \circ H$. Since S is an inverse perfect secure dominating set of $G \circ H$ with respect to D, it follows that a subset $C = D \cup S$ of $V(G \circ H)$is a disjoint perfect secure dominating set of $G \circ H$ with respect to D.■

Lemma 2.17. Let G and H be connected nontrivial graphs. If $S = V(G)$ and $D = \bigcup_{v\in S} D_v$ where $D_v = \{x\}$ is a secure dominating set of H^v for each $v \in S$, then a subset $C = D \cup S$ of $V(G \circ H)$ is a disjoint perfect secure dominating set of $G \circ H$ with respect to D.

Proof: $S = V(G)$ is a perfect secure dominating set of $G \circ H$ following the proof of Lemma 2.16. Now, $D \subseteq V(G \circ H) \setminus S = \bigcup_{v\in S} V(H^v)$. Let $D_v \subset V(H^v)$ such that $D = \bigcup_{v\in S} D_v$ where $D_v = \{x\}$ is a secure dominating set of H^v for each $v \in S$. By similar proof of Lemma 2.16., D is a perfect secure dominating set of $G \circ H$. Since $|D| = \left|\bigcup_{v\in S}\{x\}\right| = |S| \cdot |\{x\}| = |S| = |V(G)|$, D is a minimum perfect secure dominating set of $G \circ H$. Thus, $S \subseteq V(G \circ H) \setminus D$ is an inverse perfect secure dominating set of $G \circ H$ with respect to D. Hence, a subset $C = D \cup S$ of $V(G \circ H)$is a disjoint perfect secure dominating set of $G \circ H$ with respect to D.■

Lemma 2.18. Let G and H be connected nontrivial graphs. If $D = \bigcup_{v\in V(G)} D_v$ where $D_v = \{x\}$ is a secure dominating set of H^v for each $v \in V(G)$ and $S = \bigcup_{v\in V(G)} S_v$ where $S_v = \{y\}$ is a secure dominating set of $H^v$for each $v \in V(G)$ with $x \neq y$, then a subset $C = D \cup S$ of $V(G \circ H)$ is a disjoint perfect secure dominating set of $G \circ H$ with respect to D.

Proof: If $D = \bigcup_{v\in V(G)} D_v$ where $D_v = \{x\}$ is a secure dominating set of H^v for each $v \in V(G)$, then D is a minimum perfect secure dominating set of $G \circ H$ with respect to D by the proof of Lemma 2.17. If $S = \bigcup_{v\in V(G)} S_v$ where $S_v = \{y\}$ is a secure dominating set of $H^v$for each $v \in V(G)$ with $x \neq y$, then S is an inverse perfect secure dominating set of $G \circ H$ with respect to D by the proof of Lemma 2.16. Thus, a subset $C = D \cup S$ of $V(G \circ H)$is a disjoint perfect secure dominating set of $G \circ H$ with respect to D.■

The next result presents a characterization of a disjoint perfect secure dominating set in the corona of two connected graphs.

Theorem 2.19. Let G and H be connected nontrivial graphs. A subset $C = D \cup S$ of $V(G \circ H)$ is a disjoint perfect secure dominating set of $G \circ H$ with respect to D if and only of one of the following statements is satisfied:

(i) $D = V(G)$ and $S = \bigcup_{v \in D} S_v$ where $S_v = \{x\}$ is a secure dominating set of H^v for each $v \in D$.
(ii) $S = V(G)$ and $D = \bigcup_{v \in S} D_v$ where $D_v = \{x\}$ is a secure dominating set of H^v for each $v \in S$.
(iii) $D = \bigcup_{v \in V(G)} D_v$ where $D_v = \{x\}$ is a secure dominating set of H^v for each $v \in V(G)$ and $S = \bigcup_{v \in V(G)} S_v$ where $S_v = \{y\}$ is a secure dominating set of H^v for each $v \in V(G)$ with $x \neq y$.

Proof: Suppose that a proper subset $C = D \cup S$ is a disjoint perfect secure dominating set of $G \circ H$ with respect to D. Then D is a minimum perfect secure dominating set of $G \circ H$ and S is an inverse perfect secure dominating set of $G \circ H$.

Case 1. Let $D = V(G)$. Since D is a secure dominating set of $G \circ H$, it follows that for each $u \in V(G \circ H) \setminus D = \bigcup_{v \in D} V(H^v)$, there exists $v \in D$ such that $uv \in E(G \circ H)$ and $(D \setminus \{v\}) \cup \{u\}$ is a dominating set of $G \circ H$. This implies that for some $v \in D, \{u\}$ is a dominating set of $V(v + H^v)$ for each $u \in V(H^v)$. Hence, for some $v \in D$, H^v is a complete subgraph of $v + H^v$ and of $G \circ H$, that is, H is a complete graph. Let $S \subseteq V(G \circ H) \setminus D$. Then $S \subseteq \bigcup_{v \in D} V(H^v)$. Let $S_v \subseteq V(H^v)$ for each $v \in D$. If $S_v = V(H^v)$ for each $v \in D$, then v is dominated by each $u \in S_v$ (more than one u), that is, S_v is not a perfect dominating set of $v + H^v$. This is contrary to the definition of $S = \bigcup_{v \in D} S_v$. Thus, $S_v \subset V(H^v)$ for each $v \in D$. Similarly, if $2 \leq |S_v| < |V(H)|$, then v is dominated by more than one $u \in V(H^v)$, that is, S is not a perfect dominating set of $G \circ H$, a contradiction. Thus, $|S_v| = 1$. Let $S_v = \{x\}$ for each $v \in D$. Since H is a complete graph, S_v is a secure dominating set of H^v for each $v \in D$. Thus, $D = V(G)$ and $S = \bigcup_{v \in D} S_v$ where $S_v = \{x\}$ is a secure dominating set of H^v for each $v \in D$, showing statement (i).

Case 2. Let $S = V(G)$. Since S is a secure dominating set of $(G \circ H)$, it follows that for each $u \in V(G \circ H) \setminus S = \bigcup_{v \in S} V(H^v)$, there exists $v \in S$ such that $uv \in E(G \circ H)$ and $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of $G \circ H$. This implies that for some $v \in S, \{u\}$ is a dominating set of $V(v + H^v)$ for each $u \in V(H^v)$. Hence, for some $v \in S$, H^v is a complete subgraph of $v + H^v$ and of $G \circ H$, that is, H is a complete graph. Let $D \subseteq V(G \circ H) \setminus S$. Then $D \subseteq \bigcup_{v \in S} V(H^v)$. Let $D_v \subseteq V(H^v)$ for each $v \in S$. If $D_v = V(H^v)$ for each $v \in S$, then v is dominated by each $u \in D_v$ (more than one u), that is, D_v is not a perfect dominating set of $v + H^v$. This is contrary to the definition of $D = \bigcup_{v \in S} D_v$. Thus, $D_v \subset V(H^v)$ for each $v \in S$. Similarly, if $2 \leq |D_v| < |V(H)|$, then v is dominated by more than one $u \in V(H^v)$, that is, D is not a perfect dominating set of $G \circ H$, a contradiction. Thus, $|D_v| = 1$. Let $D_v = \{x\}$ for each $v \in S$. Since H is a complete graph, D_v is a secure dominating set of H^v for each $v \in S$. Thus, $S = V(G)$ and $D = \bigcup_{v \in S} D_v$ where $D_v = \{x\}$ is a secure dominating set of H^v for each $v \in S$, showing statement (ii).

Case 3. Let $D \neq V(G)$ and $S \neq V(G)$. Then $D \subseteq V(G \circ H) \setminus V(G) = \bigcup_{v \in V(G)} V(H^v)$ and $S \subseteq V(G \circ H) \setminus V(G) = \bigcup_{v \in V(G)} V(H^v)$.

Let $D = \bigcup_{v \in V(G)} D_v$ where $D_v \subseteq V(H^v)$ for each $v \in V(G)$. If $D_v = V(H^v)$ for each $v \in V(G)$, then v is dominated by each $u \in D_v$ (more than one u), that is, D_v is not a perfect dominating set of $v + H^v$. This is contrary to the definition of $D = \bigcup_{v \in V(G)} D_v$. Thus, $D_v \subset V(H^v)$ for each $v \in V(G)$. Now, if $2 \leq |D_v| < |V(H)|$, then v is dominated by more than one $u \in V(H^v)$, that is, D is not a perfect dominating set of $G \circ H$, a contradiction. Thus, $|D_v| = 1$. Let $D_v = \{x\}$ for each $v \in V(G)$. Since $D = \bigcup_{v \in V(G)} D_v$ is a secure dominating set of $G \circ H$, it follows that for each $u \in V(v + H^v) \setminus D_v, ux \in E(v + H^v)$ and $(D_v \setminus \{x\}) \cup \{u\} = \{u\}$ is a dominating set of $v + H^v$ for each $v \in V(G)$. This implies that H^v is a complete subgraph of $G \circ H$, that is, H is a complete graph. Since H is a complete graph, D_v is a secure dominating set of H^v for each $v \in V(G)$. Thus, $D = \bigcup_{v \in V(G)} D_v$ where $D_v = \{x\}$ is a secure dominating set of H^v for each $v \in V(G)$.

Further, let $S = \bigcup_{v \in V(G)} S_v$ where $S_v \subseteq V(H^v)$ for each $v \in V(G)$. If $S_v = V(H^v)$ for each $v \in V(G)$, then v is dominated by each $u \in S_v$ (more than one u), that is, S_v is not a perfect dominating set of $v + H^v$. This is contrary to the definition of $S = \bigcup_{v \in V(G)} S_v$. Thus, $S_v \subset V(H^v)$ for each $v \in V(G)$. Now, if $2 \leq |S_v| < |V(H)|$, then v is dominated by more than one $u \in V(H^v)$, that is, S is not a perfect dominating set of $G \circ H$, a contradiction. Thus, $|S_v| = 1$. Let $S_v = \{y\}$ for each $v \in V(G)$. Since $S = \bigcup_{v \in V(G)} S_v$ is a secure dominating set of $G \circ H$, it follows that for each $u \in V(v + H^v) \setminus S_v, ux \in E(v + H^v)$ and $(S_v \setminus \{x\}) \cup \{u\} = \{u\}$ is a dominating set of $v + H^v$ for each $v \in V(G)$. This implies that H^v is a complete subgraph of $G \circ H$, that is, H is a complete graph. Since H is a complete graph, S_v is a secure dominating set of H^v for each $v \in V(G)$. Thus $S = \bigcup_{v \in V(G)} S_v$ where $S_v = \{y\}$ is a secure dominating set of H^v for each $v \in V(G)$ with $D_v \cap S_v = \emptyset$, that is $x \neq y$.

Therefore, $D = \bigcup_{v \in V(G)} D_v$ where $D_v = \{x\}$ is a secure dominating set of H^v for each $v \in V(G)$ and $S = \bigcup_{v \in V(G)} S_v$ where $S_v = \{y\}$ is a secure dominating set of H^v for each $v \in V(G)$ with $x \neq y$, showing statement (iii).

For the converse, suppose that statement (i) is satisfied. Then $D = V(G)$ and $S = \bigcup_{v \in D} S_v$ where $S_v = \{x\}$ is a secure dominating set of H^v for each $v \in D$. By Lemma 2.16., a subset $C = D \cup S$ of $V(G \circ H)$ is a disjoint perfect secure dominating set of $G \circ H$ with respect to D.

Suppose that statement (ii) is satisfied. Then $S = V(G)$ and $D = \bigcup_{v \in S} D_v$ where $D_v = \{x\}$ is a secure dominating set of H^v for each $v \in S$. By Lemma 2.17, a subset $C = D \cup S$ of $V(G \circ H)$ is a disjoint perfect secure dominating set of $G \circ H$ with respect to D.

Suppose that statement (iii) is satisfied. Then $D = \bigcup_{v \in V(G)} D_v$ where $D_v = \{x\}$ is a secure dominating set of H^v for each $v \in V(G)$ and $S = \bigcup_{v \in V(G)} S_v$ where $S_v = \{y\}$ is a secure dominating set of H^v for each $v \in V(G)$ with $x \neq y$. By Lemma 2.18, a subset $C = D \cup S$ of $V(G \circ H)$ is a disjoint perfect secure dominating set of $G \circ H$ with respect to D. ■

The following result is an immediate consequence of Theorem 2.19.

Corollary 2.20. Let $G$and H be connected graphs. Then

$$\gamma_{ps}\gamma_{ps}(G \circ H) = 2 \cdot \gamma(G \circ H)$$

if and only if H is a complete graph.

Proof: Suppose that the $\gamma_{ps}\gamma_{ps}(G \circ H) = 2 \cdot \gamma(G \circ H)$. Since $\gamma(G \circ H) = |V(H)|$ by Remark 2.14, $\gamma_{ps}\gamma_{ps}(G \circ H) = 2 \cdot |V(G)|$. Let $C = D \cup S$ be a $\gamma_{ps}\gamma_{ps}$-set in $G \circ H$. In view of Theorem 2.19. *(i)*, $D = V(G)$ and $S = \bigcup_{v \in D} S_v$ where $S_v = \{x\}$ is a secure dominating set of H^v for each $v \in D$, that is, $S_v = \{x\}$ is a perfect secure dominating set of H^v for each $v \in D$. By Remark 2.6, H is a complete graph.

For the converse, suppose that H is a complete graph. Suppose that $D = V(G)$. Then D is a minimum perfect secure dominating set of $G \circ H$by Remark 2.14 and since H is a complete graph. Let $S \subset V(G \circ H) \setminus D = \bigcup_{v \in D} V(H^v)$ and suppose that $S = \bigcup_{v \in D} S_v$, where $S_v \subseteq V(H^v)$ for each $v \in D$. Since H is a complete graph, $\gamma(H) = 1$. Let $S_v = \{x\}$ for each $v \in D$. Then for each $v \in D$, S_v is a secure dominating set of H^v. By Theorem 2.19 *(i)*, a subset $C = D \cup S$ of $V(G \circ H)$ is a disjoint perfect secure dominating set of $G \circ H$ with respect to D. Thus,

$$\begin{aligned}\gamma_{ps}\gamma_{ps}(G \circ H) &\leq |C| \\ &= |D \cup S| \\ &= \left| V(G) \cup \left(\bigcup_{v \in D} S_v \right) \right|\end{aligned}$$

$$
\begin{aligned}
&= |V(G)| + \left|\left(\bigcup_{v \in D} S_v\right)\right| \\
&= |V(G)| + \sum_{v \in D} |S_v| \\
&= |V(G)| + |D| \cdot |S_v| \\
&= |V(G)| + |V(G)| \cdot 1 \\
&= 2 \cdot |V(G)| \\
&= 2 \cdot \gamma(G \circ H),
\end{aligned}
$$

that is, $\gamma_{ps}\gamma_{ps}(G \circ H) \leq 2 \cdot \gamma(G \circ H)$. Since $\gamma_{ps}\gamma_{ps}(G \circ H) \neq |V(G)|$ by Remark 2.15, it follows that $\gamma_{ps}\gamma_{ps}(G \circ H) \geq 2 \cdot |V(G)|$. Thus,

$$2 \cdot \gamma(G \circ H) = 2 \cdot |V(G)| \leq \gamma_{ps}\gamma_{ps}(G \circ H) \leq 2 \cdot \gamma(G \circ H),$$

that is, $\gamma_{ps}\gamma_{ps}(G \circ H) = 2 \cdot \gamma(G \circ H)$. ■

III. CONCLUSION AND RECOMMENDATIONS

In this paper, we introduced a new parameter of domination of graphs — the disjoint perfect secure domination in graphs. We characterized this parameter in the join and corona of graphs, and determined their exact values. This study will open new areas for research relevant to the concept of combining different variants of domination. In addition, further study on other binary operations of graphs involving disjoint perfect secure domination is possible for expanding the understanding of this new parameter. Moreover, the disjoint perfect secure domination of the Cartesian product and the lexicographic product of graphs is a potential topic for future research.

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