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A note on the existence of regular r-partite self-complementary and almost self-complementary r-uniform hypergraphs

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Abstract: A hypergraph H is said to be r-partite r-uniform if its vertex set V can be partitioned into non-empty sets $V_1, V_2, \cdots, V_r$ so that every edge in the edge set $E(H)$, consists of precisely one vertex from each set V_i, $i = 1,2,\cdots,r$. It is denoted as $H^r(V_1,V_2,\cdots,V_r)$ or $H^r_{(n_1,n_2,\cdots,n_r)}$ if $|V_i| = n_i$ for $i = 1,2,\cdots,r$. In this paper, we prove a necessary condition for the existence of r-partite self-complementary r-uniform hypergraph. Further, we prove that there does not exist a regular r-partite almost self-complementary r-uniform hypergraph.

Keywords: r-partite r-uniform hypergraph, r-partite self-complementary r-uniform hypergraph, r-partite almost self-complementary r-uniform hypergraph

Mathematics Subject Classification: 05C65.

1 Introduction

Let V be a finite set with n vertices. By $\binom{V}{k}$ we denote the set of all k-subsets of V. A k-uniform hypergraph is a pair $H = (V;E)$, where $E \subset \binom{V}{k}$. V is called a vertex set, and E an edge set of H. Two k-uniform hypergraphs $H = (V;E)$ and $H' = (V';E')$ are isomorphic if there is a bijection $\sigma : V \to V'$ such that σ induces a bijection of E onto E'. If $H = (V;E)$ is isomorphic to $H' = \left(V;\binom{V}{k} - E\right)$, then H is called a self-complementary k-uniform hypergraph. Every permutation $\pi : V \to V$ which induces a bijection $\pi' : E \to \binom{V}{k} - E$ is called a self-complementing permutation.

A. Syman'ski, A. P. Wojda ([11],[12],[13]) and S. Gosselin [3], independently characterized n and k for which there exist k-uniform self-complementary hypergraphs of order n.

P. Poto$\breve{c}$nik and M. $\breve{S}$ajana [10] proved the existence of regular self-complementary 3-uniform hypergraphs. S. Gosselin [4] has characterized all n and k for which there exists a regular k-uniform self-complementary hypergraph of order n.

In [5], the existence of an almost self-complementary 3-uniform hypergraphs is proved and in [14], Wojda generalized these results for k-uniform hypergraphs.

T. Gangopadhyay and S. P. Rao Hebbare [2] studied bipartite self-complementary graphs. In [6], a bipartite self-complementary 3-uniform hypergraph H with partition (V_1,V_2) of a vertex set V such that $|V_1| = m$ and $|V_2| = n$ is studied. In [7], Kamble L. N. and et al. found the conditions on m and n for a bipartite self-complementary 3-uniform hypergraph $H^3(V_1,V_2)$ to be regular and quasi-regular. In [9], bipartite almost self-complementary 3-uniform hypergraphs are studied. In [8], r-partite self-complementary and almost self-complementary r-uniform hypergraphs are studied.

In this paper, we prove a necessary condition for the existence of regular r-partite self-complementary r-uniform hypergraph. We prove there does not exist a regular r-partite almost self-complementary r-uniform hypergraph. We further prove that there does not exist a quasi-regular r-partite self-complementary r-uniform hypergraph.

2 Preliminary definitions and results

Definition 2.1 (r-partite r-uniform Hypergraph)[1] *A hypergraph H is said to be r-partite r-uniform if its vertex set V can be partitioned into non-empty sets $V_1, V_2, \cdots, V_r$ so that every edge in the edge set $E(H)$, consists of precisely one vertex from each set V_i, $i = 1,2,\cdots,r$.*

An r-partite r-uniform hypergraph is denoted as $H^r(V_1, V_2, \cdots, V_r)$ or $H^r_{(n_1,n_2,\cdots,n_r)}$ if $|V_i| = n_i$ for $i = 1,2,\cdots,r$. A 2-partite 2-uniform hypergraph is nothing but a bipartite graph.

Definition 2.2 (Complete r-partite r-uniform hypergraph)[1] *An r-partite r-uniform hypergraph H with the vertex set $V = \bigcup_{i=1}^{r} V_i, V_i \cap V_j = \varphi, \forall\, i \neq j$ and the edge set $E = \{e : e \subset V, |e| = r$ and $e \cap V_i \neq \phi,$ for $i = 1,2,\cdots,r\}$ is called* **a complete r-partite r-uniform hypergraph**.

A complete r-partite r-uniform hypergraph is denoted as $K^r(V_1, V_2, \cdots, V_r)$ or $K^r_{(n_1,n_2,\cdots,n_r)}$ if $|V_i| = n_i$ for $i = 1,2,\cdots,r$. We observe that, the total number of edges in $K^r_{(n_1,n_2,\cdots,n_r)}$ is $\prod_{i=1}^{r} n_i$.

Given an r-partite r-uniform hypergraph $H = H^r(V_1, V_2, \cdots, V_r)$, its r**-partite complement** is the r-partite r-uniform hypergraph $\overline{H} = \overline{H}^r(V_1, V_2, \cdots, V_r)$ where $V(\overline{H}) = V(H)$ and $E(\overline{H}) = E(K^r(V_1, V_2, \cdots, V_r)) - E(H)$.

$\overline{H}$ is said to the complement of H with respect to $K^r(V_1, V_2, \cdots, V_r)$. An r-partite r-uniform hypergraph $H = H^r(V_1, V_2, \cdots, V_r) = H^r(V)$ is said to be **self-complementary** if it is isomorphic to its r-partite complement $\overline{H} = \overline{H}^r(V_1, V_2, \cdots, V_r) = \overline{H}^r(V)$, that is there exists a bijection $\sigma : V \to V$ such that e is an edge in H if and only if $\sigma(e)$ is an edge in $\overline{H}$.

We use the shortform "r-psc" for r-partite self-complementary r-uniform hypergraph.

Following result gives a necessary and sufficient condition for the existence of r-partite self-complementary r-uniform hypergraph.

Result 2.3 [8] *There exists an r-psc $H^r(V_1, V_2, \cdots, V_r)$ where $|V_i| = n_i$ for $i = 1,2,\cdots,r$ if and only if at least one of $n_1, n_2, \cdots, n_r$ is even.*

It is clear that the partitioning of the edge set of $K^r(V_1, V_2, \cdots, V_r)$ into two isomorphic factors is not possible when $K^r(V_1, V_2, \cdots, V_r)$ has an odd number of edges. However, after deleting some odd number of edges from $K^r(V_1, V_2, \cdots, V_r)$ the remaining r-uniform hypergraph may be partitioned into two isomorphic factors. By deleting one edge from $K^r(V_1, V_2, \cdots, V_r)$ an almost complete r-partite r-uniform hypergraph is defined in [8] as follows.

Definition 2.4 (Almost complete r-partite r-uniform hypergraph)[8] *The hypergraph $\tilde{K}^r_{(n_1,n_2,\cdots,n_r)} = K^r_{(n_1,n_2,\cdots,n_r)} - e$ is called an almost complete r-partite r-uniform hypergraph where e is an edge in $K^r_{(n_1,n_2,\cdots,n_r)}$ called the deleted edge. Vertices of e will be called the special vertices.*

Definition 2.5 (r-partite almost self-complementary r-uniform hypergraph)[8] *An r-partite r-uniform hypergraph $H(V_1, V_2, \cdots, V_r)$ such that $|V_i| = n_i$ for $i = 1,2,\cdots,r$ is almost self-complementary if it is isomorphic with its complement $\overline{H}(V_1, V_2, \cdots, V_r)$ with respect to $\tilde{K}^r_{(n_1,n_2,\cdots,n_r)}$.*

We use the shortform “ r -pasc" for r -partite almost self-complementary r -uniform hypergraph.

Following result gives a necessary and sufficient condition for the existence of r -pasc.

Result 2.6 *[8] There exists an r -pasc $H^r(V_1,V_2,\cdots,V_r)$ where $|V_i|=n_i$ for $i=1,2,\cdots,r$ if and only if $n_1,n_2,\cdots,n_r$ are odd.*

Definition 2.7 (Degree of a vertex) *The degree of a vertex v in a hypergraph H is the number of edges containing the vertex v and is denoted as $d_H(v)$.*

Definition 2.8 (Regular hypergraph) *A hypergraph H is said to be regular if all vertices have the same degree.*

Definition 2.9 (Quasi-regular hypergraph) *A hypergraph H is said to be quasi-regular if the degree of each vertex is either r or $r-1$ for some positive integer r.*

3 Necessary condition for the existence of a regular r -partite self-complementary r -uniform hypergraphs

It is known that [10], a regular self-complementary 3-uniform hypergraph on n vertices exists if and only if n is congruent to 1 or 2 modulo 4. In [7], it is proved that there exists a regular bipartite self-complementary 3-uniform hypergraph $H(V_1,V_2)$ with $|V_1|=m,|V_2|=n,m+n>3$ if and only if $m=n$ and n is congruent to 0 or 1 modulo 4.

We know that every edge of $K^r(V_1,V_2,\cdots,v_r)=K^r_{(n_1,n_2,\cdots,n_r)}$ contains exactly one vertex from each set $V_i, i=1,2,\cdots,r$. And for any $u_i\in V_i, i=1,2,\cdots,r$ in $K^r(V_1,V_2,\cdots,V_r)=K^r_{(n_1,n_2,\cdots,n_r)}$, the degree of u_i is $\prod_{\substack{j=1\\ j\neq i}}^{r} n_j$.

In the following theorem, we prove a necessary condition for the existence of a regular r -partite self-complementary r -uniform hypergraph.

Theorem 3.1 *If there exists a regular r -psc $H^r(V_1,V_2,\cdots,V_r)$ where $|V_i|=n_i$ for $i=1,2,\cdots,r$, then all $n_1,n_2,\cdots,n_r$ are equal and even.*

Proof. Suppose there exists a regular r -psc $H^r(V_1,V_2,\cdots,V_r)$ where $|V_i|=n_i$ for $i=1,2,\cdots,r$ with complementing permutation σ. Let r be its regular degree.

For any vertex $u_i\in V_i, i=1,2,\cdots,r$, we have

$$d_H(u_i)+d_H(\sigma(u_i)) = \text{Degree of } u_i \text{ in } K^r_{(n_1,n_2,\cdots,n_r)}.$$

That is

$$d_H(u_i)+d_H(\sigma(u_i)) = \prod_{\substack{k=1\\ k\neq i}}^{r} n_k.$$

That is

$$r+r = \prod_{\substack{k=1\\ k\neq i}}^{r} n_k.$$

That is

$$2r = \prod_{\substack{k=1 \\ k\neq i}}^{r} n_k. \quad (1)$$

Similarly, for any $u_j \in V_j,\ j = 1,2,\cdots,r,\ j \neq i$,

$$d_H(u_j) + d_H(\sigma(u_j)) = \text{Degree of } u_j \text{ in } K^r_{(n_1,n_2,\cdots,n_r)}.$$

$$d_H(u_j) + d_H(\sigma(u_j)) = \prod_{\substack{k=1 \\ k\neq j}}^{r} n_k.$$

That is

$$r + r = \prod_{\substack{k=1 \\ k\neq j}}^{r} n_k.$$

That is

$$2r = \prod_{\substack{k=1 \\ k\neq j}}^{r} n_k. \quad (2)$$

From equations (1) and (2) we get that

$$2r = \prod_{\substack{k=1 \\ k\neq i}}^{r} n_k = \prod_{\substack{k=1 \\ k\neq j}}^{r} n_k. \quad (3)$$

Solving above equation we get that $n_i = n_j,\ \forall\, i, j = 1,2,\cdots,r$. Since i, j are arbitrary we get that $n_1 = n_2 = \cdots = n_r$ and all are even.

In the next theorem, we prove that there does not exist a quasi-regular r-psc.

Theorem 3.2 *There does not exist a quasi-regular r-psc.*

Proof. Suppose there exists a quasi-regular r-psc $H^r(V_1,V_2,\cdots,V_r)$ where $|V_i| = n_i$ for $i = 1,2,\cdots,r$ with complementing permutation σ. Since $H^r(V_1,V_2,\cdots,V_r)$ is quasi-regular there exist $u_i \in V_i$, such that $d_H(u_i) \neq d_H(\sigma(u_i))$ for some $i, i = 1,2,\cdots,r$. Suppose $d_H(u_i) = r$ and $d_H(\sigma(u_i)) = r-1$.

We have $d_H(u_i) + d_H(\sigma(u_i)) = \text{Degree of } u_i \text{ in } K^r_{(n_1,n_2,\cdots,n_r)}$.

That is $r + r - 1 = \prod_{\substack{k=1 \\ k\neq i}}^{r} n_k$, which is even, a contradiction.

Theorem 3.3 *There does not exist a regular r-pasc.*

Proof. Suppose there exists a regular r-partite almost self-complementary r-uniform hypergraph $H^r(V_1,V_2,\cdots,V_r)$ where $|V_i| = n_i$ for $i = 1,2,\cdots,r$ with complementing permutation σ. From theorem 2.6, it is obvious that $n_1, n_2, \cdots, n_r$ are all odd.

Let $u_i \in V_i$ for any $i, i = 1,2,\cdots,r$. We have

$$d_H(u_i) + d_H(\sigma(u_i)) = \text{the degree of } u_i \text{ in } \tilde{K}^r(V_1,V_2,\cdots,V_r). \qquad (4)$$

If u_i is not a special vertex then equation (4) becomes $r + r = \prod_{\substack{k=1 \\ k \neq i}}^{r} n_k$.

That is

$$2r = \prod_{\substack{k=1 \\ k \neq i}}^{r} n_k. \qquad (5)$$

And if u_i is a special vertex then equation (4) becomes $r + r = \prod_{\substack{k=1 \\ k \neq i}}^{r} n_k - 1$.

That is

$$2r = \prod_{\substack{k=1 \\ k \neq i}}^{r} n_k - 1. \qquad (6)$$

From equations (5) and (6), we get a contradiction.

References

[1]. R. Diestel, *Graph Theory* (3rd edition). Springer-Verlag, 2005.

[2]. T. Gangopadhyay and S. P. Rao Hebbare, *Structural properties of r-partite complementing permutations*, Tech. Report No. 19/77, I.S.I, Calcutta.

[3]. S. Gosselin, *Constructing self-complementary uniform hypergraphs*, Discrete Math., 310(2010), 1366-1372.

[4]. S. Gosselin, *Constructing regular self-complementary uniform hypergraphs*, Combinatorial Designs, 16(6) (2011), 439-454.

[5]. L.N. Kamble, C.M. Deshpande and B.Y. Bam, *Almost self-complementary 3-uniform hypergraphs*, Discuss. Math. Graph Theory 37 (2017) 131-140.

[6]. L.N. Kamble, C.M. Deshpande, B.P. Athawale, *On self-complementary bipartite 3-uniform hypergraph*, Ars. Combin. **146**(2019), 293–305..

[7]. L.N. Kamble, C. M. Deshpande, B.P. Athawale, *The existence of regular and quasi-regular bipartite self-complementary 3-uniform hypergraphs*, J. Comb. Math. Comb. Comput. **111** (2019), 257–268.

[8]. L.N. Kamble, C.M. Deshpande, B.P. Athawale, *r-partite self-complementary and almost self-complementary r-uniform hypergraphs*, AKCE Int. J. Graphs Comb. **17(1)** (2020), 159–167.

[9]. L.N. Kamble, C.M. Deshpande, B.P. Athawale, *The existence of bipartite almost self-complementary 3-uniform hypergraphs*, Opuscula Math. **43(5)** (2023), 663â€“673.

[10]. P. Poto$\breve{c}$nik, M. $\breve{S}$ajana, *The existence of regular self-complementary 3-uniform hypergraphs*, Discrete Math., 309 (2009), 950-954.

[11]. A. Szyman'ski, A. P. Wojda, *A note on k-uniform self-complementary hypergraphs of given order*, Discuss. Math. Graph Theory 29(2009), 199-202.

[12]. A. Szyman'ski, A. P. Wojda, *Self-complementing permutations of k-uniform hypergraphs*, Discrete Math. and Theoretical Computer Science 11(2009), 117-124.

[13]. A. Wojda, *Self complementary hypergraphs*, Discuss. Math. Graph Theory 26(2006), 217-224.

[14]. A. Wojda, *Almost self-complementary uniform hypergraphs*, Discuss. Math. Graph Theory 38(2018), 607-610.

