An Overview on Self-Complementary 3-Uniform Hypergraphs

Lata N. Kamble

Department of Mathematics, Abasaheb Garware College, Pune-411004, India

Abstract: Hypergraph is a generalization of a graph in which an edge may contain more than two vertices. A hypergraph H is called a k-uniform (or a k-uniform hypergraph) if every edge contains exactly k number of vertices. In this paper, we review the current research in self-complementary 3-uniform hypergraphs and bipartite self-complementary 3-uniform hypergraphs. We mainly focus on necessary and sufficient conditions on the order for existence and the cycle structure of complementing permutations for each of these self-complementary classes.

Keywords: hypergraph, self-complementary, complementing permutation, quasi-regular

Mathematics Subject Classification: 05C65.

1. Introduction

Hypergraph is a generalization of a graph in which an edge may contain more than two vertices. The theory of hypergraphs is popularized and enriched by many contributions by C. Berge [1, 2]. Hypergraphs are a generalization of graphs and have been studied along the lines of graphs. Researchers have worked with uniform hypergraphs that are self-complementary. Most of the research in self-complementary uniform hypergraphs is about determining necessary and sufficient conditions on the order of these structures.

In this paper, we review the current research in self-complementary 3-uniform hypergraphs. We concentrate on self-complementary 3-uniform hypergraphs and bipartite self-complementary 3-uniform hypergraphs. We mainly focus on necessary and sufficient conditions on order for the existence of these classes and the cycle structure of complementing permutations for each of these self-complementary classes. We also explore the results on existence of regular and quasi-regular self-complementary 3-uniform hypergraphs and bipartite self-complementary 3-uniform hypergraphs.

2. Basic Concepts

Definition 2.1 (Hypergraph) *A hypergraph H with vertex set V and edge set E is a pair (V,E), in which V is a finite nonempty set and E is a collection of nonempty subsets of V.*

For a finite set V and a positive integer k, let $V^{(k)}$ denote the set of all k-subsets of V.

Definition 2.2 (k-uniform hypergraph) *A hypergraph (V,E) is called k-uniform (or a k-uniform hypergraph) if E is a subset of $V^{(k)}$.*

The parameters k and $|V|$ are called the rank and the order of the k-uniform hypergraph, respectively. The vertex set and the edge set of a hypergraph H will often be denoted by $V(H)$ and $E(H)$, respectively. A 2-uniform hypergraph is a graph. In this paper, we focus on 3-uniform hypergraphs.

Definition 2.3 (Degree of a vertex) *The degree of a vertex v in a hypergraph H is the number of edges containing the vertex v and is denoted as $d_H(v)$.*

Definition 2.4 (Regular hypergraph) *A hypergraph H is said to be regular if all its vertices have the same degree.*

Definition 2.5 *An isomorphism between two k-uniform hypergraphs H and H' is a bijection $\sigma: V(H) \to V(H')$ which induces a bijection from $E(H)$ to $E(H')$.*

If such an isomorphism exists, the hypergraphs H and H' are said to be isomorphic.

Definition 2.6 *(* **Complete k-uniform hypergraph***) A complete k-uniform hypergraph is the k-uniform hypergraph, K_n^k on n vertices whose edge set is $V^{(k)}$.*

Definition 2.7 (Complement of a k-uniform hypergraph) *The complement $\overline{H}$ of a k-uniform hypergraph $H(V,E)$ is the hypergraph with vertex set V and edge set $\overline{E} = V^{(k)} \setminus E$.*

We also say that $\overline{H}$ is the complement of H with respect to K_n^k.

Definition 2.8 (Self-complementary k-uniform hypergraph) *A k-uniform hypergraph H is called self-complementary if it is isomorphic to its complement $\overline{H}$.*

Thus H is self-complementary if and only if there exists a bijection $\sigma : V \to V$ such that e is an edge in H if and only if $\sigma(e)$ is an edge in $\overline{H}$. Such a σ is called a **complementing permutation**.

In 1960, 2-uniform self-complementary hypergraphs (i.e graphs) were first studied independently by G. Ringel [9] and H. Sach [11]. They determined necessary and sufficient conditions on the order n of a self-complementary graph (2-hypergraph). They proved that, "A 2-hypergraph $G(V,E)$ on n vertices is self-complementary if and only if n is congruent to 0 or 1 modulo 4." They also characterized the cycle structure of complementing permutation of self-complementary graphs.

The following is a well-known result regarding the cycle structure of complementing permutation of self-complementary graphs (2-hypergraphs) proved by Ringel [9] and Sachs [11]. This result is also proved by Suprunenko [13].

Result 2.9 *σ is a complementing permutation of a self-complementary graph G if and only if one of the following holds:*

(i). Every cycle of σ is of length a multiple of 4.

(ii). σ has exactly one fixed point and all the other cycles of σ are of length multiple of 4.

In 1985, Suprunenko [13] generalized the method of Ringel and Sachs for generating all self-complementary graphs to find a method for generating all self-complementary 3-uniform hypergraphs. His characterization of the cycle structure of complementing permutations of a 3-uniform hypergraph was also proved independently by Kocay [7]. They proved the following analogous result for 3-uniform hypergraphs.

Result 2.10 *[7] σ is a complementing permutation of a self-complementary 3-uniform hypergraph if and only if either*

(i). every cycle of σ has even length

or

(ii) σ has 1 or 2 fixed points and all the other cycles of σ have length a multiple of 4.

In 2007, Szyman'ski and Wojda [14] solved the problem of the existence of a self-complementary k-uniform hypergraph of order n. They proved the following result.

Result 2.11 *[14] Let k and n be positive integers such that $k \le n$. A self-complementary k-uniform hypergraph of order n exists if and only if $\binom{n}{k}$ is even.*

For a self-complementary 3-uniform hypergraph, the Result 2.11 can be restated as

Result 2.12 *[14] A self-complementary 3-uniform hypergraph of order n exists if and only if $n \equiv 0$ or 1 or $2 (\mathrm{mod}\, 4)$.*

In 2007, Poto$\breve{c}$nik and $\breve{S}$ajna [10] gave constructions for regular self-complementary 3-hypergraphs of every admissible order n (that is, $n \equiv 1$ or $2 (\bmod 4)$). Authors proved the following result on existence of regular self-complementary 3-uniform hypergraphs.

Theorem 2.13 *[10] There exists a regular self-complementary 3-uniform hypergraph of order n if and only if $n \geq 5$ and n is congruent to 1 or 2 modulo 4.*

3. Quasi-regular and Bi-regular Self-complementary 3-uniform Hypergraphs

We begin with the definition of bi-regular and quasi-regular hypergraph and then state the results on existence of bi-regular and quasi-regular 3-uniform hypergraphs.

Definition 3.1 (Bi-regular hypergraph)*[3] A hypergraph H is said to be bi-regular if there exist two distinct positive integers d_1 and d_2 such that the degree of each vertex is either d_1 or d_2 and there is at least one vertex each with degree d_1 and d_2.*

Definition 3.2 (Quasi-regular hypergraph) *[3] A hypergraph H is said to be quasi-regular if the degree of each vertex is either r or $r-1$ for some positive integer r such that there is at least one vertex each with degree r and $r-1$.*

It is clear that every quasi-regular hypergraph is bi-regular but not conversely.

Kamble L. N. and et.al [3], proved the existence of bi-regular and quasi-regular self-complementary 3-uniform hypergraphs.

The next lemma gives the non-existence of bi-regular self-complementary 3-uniform hypergraph.

Lemma 3.3 *[3] If H is a self-complementary 3-uniform hypergraph on n vertices where $n \equiv 1 (\bmod 4)$ and $n \geq 5$ then H cannot be bi-regular.*

Obviously there does not exist a quasi regular self-complementary 3-uniform hypergraph on n vertices if $n \equiv 1 (\bmod 4)$.

The following theorem gives a necessary and sufficient condition on the order n for the existence of quasi-regular self-complementary 3-uniform hypergraph.

Theorem 3.4 *[3] There exists a quasi-regular self-complementary 3-uniform hypergraph of order n if and only if $n \geq 4$ and $n \equiv 0 (\bmod 4)$.*

In the next theorem a necessary and sufficient condition on the order n of a self-complementary 3-uniform hypergraph to be bi-regular is given.

Theorem 3.5 *[3] There exists a bi-regular self-complementary 3-uniform hypergraph of order n if and only if either $n \equiv 0 (\bmod 4)$ or $n \equiv 2 (\bmod 4)$ and $n \geq 4$.*

In the next section we explore the results on bipartite self-complementary 3-uniform hypergraphs.

4. Bipartite Self-complementary 3-uniform Hypergraphs

Definition 4.1 *(* **Bipartite hypergraph***) A hypergraph H with vertex set V and edge set E is called bipartite if V can be partitioned into two subsets V_1 and V_2 such that $e \cap V_1 \neq \phi$ and $e \cap V_2 \neq \phi$ for any $e \in E$.*

A bipartite k-uniform hypergraph is a 3-uniform hypergraph $H^k(V_1, V_2)$ on $V_1 \cup V_2$ vertices such that $V_1 \cap V_2 = \phi$ and every edge of H has nonempty intersection with V_1 as well as V_2. In particular when $k = 3$, $H^3(V_1, V_2)$ is a bipartite 3-uniform hypergraph such that every edge of $H^3(V_1, V_2)$ contains one vertex from one part and two vertices from the other part of the partition V_1 and V_2 of V. A triple $\{x, y, z\}$ of vertices such that x, y, z belong to a single part of the partition of V is not an edge of $H^3(V_1, V_2)$. Henceforth we call it an invalid edge or an invalid triple.

Definition 4.2 *(* **Complete Bipartite 3-uniform hypergraph***) A 3-uniform hypergraph H with the vertex set $V = V_1 \cup V_2, V_1 \cap V_2 = \phi$ and the edge set $E = \{e : e \subset V, |e| = 3$ and $e \cap V_i \neq \phi,$ for $i = 1,2\}$ is called a complete bipartite 3-uniform hypergraph. It is denoted as $K^3(V_1,V_2)$ or $K^3_{(m,n)}$ where $|V_1| = m$ and $|V_2| = n$.*

A complete bipartite 3-uniform hypergraph is denoted as $K^3(V_1,V_2)$ or $K^3_{(m,n)}$. Every valid triple of vertices in V is an edge of $K^3_{(m,n)}$ which will be called as a valid edge. Hence the total number of edges in $K^3_{(m,n)}$ is $m\binom{n}{2} + n\binom{m}{2} = \frac{mn(m+n-2)}{2}$.

Definition 4.3 *(* **Complement of a bipartite 3-uniform hypergraph***) Given a bipartite 3-uniform hypergraph $H = H^3(V_1,V_2)$, we define its complement with respect to the partition (V_1,V_2) to be the 3-uniform hypergraph $\overline{H} = \overline{H}^3(V_1,V_2)$ where $V(\overline{H}) = V(H)$ and $E(\overline{H}) = E(K^3(V_1,V_2)) - E(H)$.*

Definition 4.4 *(* **Bipartite self-complementary 3-uniform hypergraph***) A bipartite 3-uniform hypergraph $H = H^3(V_1,V_2)$ is said to be bipartite self-complementary 3-uniform hypergraph if it is isomorphic to its complement $\overline{H} = \overline{H}^3(V_1,V_2)$, that is there exists a bijection $\sigma : V \to V$ such that e is an edge in H if and only if $\sigma(e)$ is an edge in $\overline{H}$, that is $\{u,v,w\}$ is an edge in H if and only if $\{\sigma(u),\sigma(v),\sigma(w)\}$ is an edge in $\overline{H}$.*

Bipartite self-complementary 3-uniform hypergraphs are studied in [4] by Kamble L. N. and et. Al.

A partitioning of the edge set of $K^3_{(m,n)}$ into two isomorphic factors is possible only if $K^3_{(m,n)}$ has an even number of edges i.e the number $\frac{mn(m+n-2)}{2}$ is even. In the following proposition, the existence of bipartite self-complementary 3-uniform hypergraph $H^3_{(m,n)}$ is given.

Theorem 4.5 *[4] There exists a bipartite self-complementary 3-uniform hypergraph $H^3_{(m,n)}$ if and only if*

either (i) $m = n$
or (ii) $m \neq n$ and either m or n is congruent to 0 modulo 4
or (iii) $m \neq n$ and both m and n are congruent to 1 or 2 modulo 4.

We observe that for any $u \in V_1$ and $v \in V_2$ in $K^3(V_1,V_2) = K^3_{(m,n)}$, the degree of u is $n(m-1) + \binom{n}{2}$ and the degree of v is $m(n-1) + \binom{m}{2}$.

Following theorem gives necessary and sufficient conditions on order of bipartite self-complementary 3-uniform hypergraph to be regular.

Theorem 4.6 *[4] There exists a regular bipartite self-complementary 3-uniform hypergraph $H(V_1,V_2)$ with $|V_1| = m, |V_2| = n, m+n > 3$ if and only if $m = n$ and n is congruent to 0 or 1 modulo 4.*

The following theorem gives necessary and sufficient conditions on the order of a bipartite self-complementary 3-uniform hypergraph to be quasi-regular.

Theorem 4.7 *[4] There exists a quasi-regular bipartite self-complementary 3-uniform hypergraph*

$H(V_1,V_2)$ with $|V_1|=m,\ |V_2|=n,\ m+n>3$ if and only if either $m=3, n=4$ or $m=n$ and n is congruent to 2 or 3 modulo 4.

4.1 Complementing Permutations Bipartite self-complementary 3-uniform hypergraphs

Given a bipartite self-complementary 3-uniform hypergraph $H^3_{(m,n)}=H$, let the edges of H be coloured red and the remaining edges of $K^3_{(m,n)}$ be coloured green. Since the two factors are isomorphic, there is a permutation σ of the vertices of $K^3_{(m,n)}$ called complementing permutation of H that induces a mapping of the red edges onto the green edges. Denote by σ' the corresponding mapping induced on the set of edges of $K^3_{(m,n)}$. Thus σ' maps each red edge onto a green edge and green edge onto a red edge. Thus every cycle of σ' is of even length. Conversely, any permutation σ on the vertex set of H which induces a mapping σ' on the edge set of $K^3_{(m,n)}$ such that every cycle of σ' is of even length is a complementing permutation of H.

Let $\xi(H,(V_1,V_2))$ be the set of all complementing permutations of the bipartite self-complementary 3-uniform hypergraph H. A cycle of a complementing permutation is said to be pure if it permutes only the vertices belonging to a single set of the partition (V_1,V_2) of the vertex set V and is said to be mixed otherwise.

We define two subclasses of $\xi(H,(V_1,V_2))$ as follows

$\xi_P(H,(V_1,V_2))=\{\sigma\in\xi(H,(V_1,V_2)),$ all cycles of σ are pure $\}$

$\xi_M(H,(V_1,V_2))=\{\sigma\in\xi(H,(V_1,V_2)),$ all cycles of σ are mixed $\}$.

If $\sigma\in\xi_P(H,(V_1,V_2))$ then σ is said to be pure and if $\sigma\in\xi_M(H,(V_1,V_2))$ then σ is said to be mixed.

Theorem 4.8 *[4] If $\sigma\in\xi(H,(V_1,V_2))$ then all the cycles of σ are either pure or mixed, that is $\xi(H,(V_1,V_2))=\xi_P(H,(V_1,V_2))\cup\xi_M(H,(V_1,V_2))$.*

From Theorem 4.8 we get that, if σ is any complementing permutation, then it is either pure or mixed. Further if σ contains a fixed vertex u that is $\sigma(u)=u$, then it must be pure.

In the remainder of the section we first see the cycle structure of pure complementing permutations followed by the cycle structure of mixed complementing permutations.

If σ is a pure complementing permutation of $H^3_{(m,n)}$, then σ can be written as $\sigma=\sigma_1\sigma_2$, where σ_1 permutes the vertices of V_1 and σ_2 permutes the vertices of V_2.

Following lemma's are used to prove a cycle structure of pure complementing permutations.

Lemma 4.9 *[4] Let σ_1 be a permutation on V_1 and σ_2 be a permutation on V_2. If either σ_1 or σ_2 has all cycles of length a multiple of 4 then $\sigma=\sigma_1\sigma_2\in\xi_P(H,(V_1,V_2))$.*

Lemma 4.10 *[4] If $\sigma=\sigma_1\sigma_2\in\xi_P(H,(V_1,V_2))$ is such that both σ_1 and σ_2 have at least one cycle whose length is not a multiple of 4 then*

(a) neither σ_1 nor σ_2 has an odd cycle of length greater than 1.

(b) neither σ_1 nor σ_2 has more than one fixed vertex.

Lemma 4.11 *[4] If $\sigma=\sigma_1\sigma_2\in\xi_P(H,(V_1,V_2))$ is such that both σ_1 and σ_2 have exactly one fixed point, then the lengths of all the remaining cycles of σ_1 and σ_2 are a multiple of 4.*

Lemma 4.12 *[4] Let σ_1 be a permutation on V_1 and σ_2 be a permutation on V_2. If each cycle of σ_1*

and σ_2 *is of even length then* $\sigma = \sigma_1\sigma_2 \in \xi_P(H,(V_1,V_2))$.

We summarize all the above results about pure complementing permutation of a bipartite self-complementary 3-uniform hypergraph in the following theorem.

Theorem 4.13 *[4] Let* σ_1 *be a permutation on* V_1 *and* σ_2 *be a permutation on* V_2. $\sigma = \sigma_1\sigma_2 \in \xi_P(H,(V_1,V_2))$ *if and only if either*

i) each cycle of σ_1 and σ_2 is of even length,

or

ii) at least one of σ_1, σ_2 has all the cycles of length a multiple of 4,

or

iii) both σ_1 and σ_2 have exactly one fixed vertex and all the other cycles of σ_1 and σ_2 are of length a multiple of 4.

In what follows, let $\sigma \in \xi_M(H,(V_1,V_2))$. Hence σ has no fixed vertex. Further every cycle of σ has length ≥ 2.

Before stating the main result about mixed complementing permutation we state several lemmas.

Lemma 4.14 *[4] If* C *is any cycle of* σ, *then* C *must be of even length.*

Lemma 4.15 *[4] If* C *is any cycle of* σ, *then* C *takes the vertices alternately from* V_1 *and* V_2.

Following remark easily follows from Lemma 4.15.

Remark 4.16 *(i) If* $\sigma \in \xi_M(H,(V_1,V_2))$ *then* $m = n$.

(ii) If $H = H^3_{(m,n)}$ is bipartite self-complementary 3-uniform hypergraph such that $m \neq n$, then any complementing permutation of H must be pure.

We summarize all the above results about mixed complementing permutations of a bipartite self-complementary 3-uniform hypergraph in the following theorem.

Theorem 4.17 *[4] If* $H^3(V_1,V_2)$ *is bipartite self-complementary* 3*-uniform hypergraph with* $|V_1| = m$ *and* $|V_2| = n$ *and* $\sigma \in \xi_M(H,(V_1,V_2))$, *then* $m = n$ *and all the cycles of* σ *are of even length. Further, if* $m = n > 2$ *then vertices in every cycle of* σ *are alternately from* V_1 *and* V_2.

References

[1]. C. Berge, *Graphs and Hypergraphs*, North- Holland, 1976.

[2]. C. Berge, *Hypergraphs: Combinatorics of Finite Sets*, North- Holland, Amsterdam, 1989.

[3]. L.N. Kamble, C.M. Deshpande and B.Y. Bam, *The existence of quasi-regular and bi-regular self-complementary* 3*-uniform hypergraphs*, Discuss. Math. Graph Theory 36 (2016) 419-426.

[4]. L.N. Kamble, C.M. Deshpande, B.P. Athawale, *On self-complementary bipartite* 3*-uniform hypergraph*, Ars. Combin. **146**(2019), 293–305..

[5]. L.N. Kamble, C. M. Deshpande, B.P. Athawale, *The existence of regular and quasi-regular bipartite self-complementary 3-uniform hypergraphs*, J. Comb. Math. Comb. Comput. **111** (2019), 257–268.

[6]. M. Knor, P. Potočnik, *A note on* 2*-subset-regular self-complementary* 3*-uniform hypergraphs*, Ars Combin. 11 (2013), 33-36.

[7]. W. Kocay, *Reconstructing graphs as subsumed graphs of hypergraphs, and some self-complementary triple systems*, Graphs Combin. 8 (1992), 259-276.

[8]. P. Potočnik, M. Šajana, *The existence of regular self-complementary* 3*-uniform hypergraphs*, Discrete Math. 309 (2009), 950-954.

[9]. G. Ringel, *Über selbstkomplementäre graphen*, Arch. Math. 14 (1963), 354-358.

[10]. P. Potočnik, M. Šajana, *The existence of regular self-complementary 3-uniform hypergraphs*, Discrete Math., 309 (2009), 950-954.

[11]. H. Sachs, *Über selbstkomplementäre graphen*, Publ. Math. Drecen, 9(1962), 270-288.

[12]. G. Shonda, *Generating self-complementary uniform hypergraphs*, Discrete Math. 310 (2010), 1366-1372.

[13]. D. A. Suprunenko, *Self-complementary graphs*, Cybernetica 21 (1985), 559-567.

[14]. A. Szymań ski, A. P. Wojda, *A note on k-uniform self-complementary hypergraphs of given order*, Discuss. Math. Graph Theory 29(2009), 199-202.

[15]. A. Szymań ski, A. P. Wojda, *Self-complementing permutations of k-uniform hypergraphs*, Discrete Math. and Theoretical Computer Science 11(2009), 117-124.

